Humanising Mathematical Education

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I Having originally chosen a title which defined a large enough space for me to feel sure that I would be able to find some place in it from which to speak when the time came, I now feel that I chose not too wisely but too well. I see it now as a powerful and evocative title that covers far more than I dreamed, and I know that I have only a tenuous grip on a few aspects of it. Perhaps it is a title that should be allocated to one lecture at each of the next ten conferences and a series of speakers invited to work on it. In this way, after ten years, we might find that *ATM* had made a contribution to the task of humanising mathematical education, the value of which no one could question.

The following are "thoughts in progress". I don't apologise for that, since no lecture has any right to be anything else, but I admit I wish the progress were more spectacular and the story accompanied by many more detailed illustrations. There is, indeed, a very great deal still to do.

Let me say, by way of background; that I don't want to catch myself acting as an evangelist or proselytiser on behalf of mathematical education. I don't expect, and I don't even want, every child to find mathematics an engrossing study, one that he wants to devote himself to either in school or in his life. Only a few will find mathematics seductive enough to sustain a long term engagement. But I *would* hope that every child could experience at a few moments in his school career (since he may not get another chance) the power and excitement of mathematics, as of every school subject, so that at the end of his formal education he at least knows what it is like and whether it is an activity that has a place in his future. Specialist teachers of mathematics, and the members of a conference like this, may easily find themselves committed to the attempt to sell mathematics to everyone. The investment of specialist teachers in others loving their subject is a more acute danger than that they are narrow and know nothing but their subject (which is manifestly not true).

It might as well be said at once that one relatively minor way to humanise mathematical education would be to stop forcing mathematics on children who have had ample opportunity to discover that it means nothing to them—at least, unless we really have some techniques that can bypass their failure and astonish them into success; but also, hopefully, arrange to make it accessible to them if and when they subsequently decide that they were mistaken.

Humanising mathematical education is not to be confused with encouraging mathematics teachers to come on like warm accepting therapists. No doubt there are occasions when this is what they have to be, though probably not as many as some teachers would like to believe. Respect for children is generally more useful to them than affectionate hand-holding; it may often need to be tough and abrasive too. But respect is not enough, either, since it doesn't necessarily carry any insight on the teacher's part into what to *do*, and it doesn't ensure the presence of the skills he ought to have in order to carry out what needs to be done if children are to be helped to learn.

In the context of teaching, affection is incomplete without respect, and respect incomplete without insight and skill. Perhaps we should remind ourselves that the teacher-learner relationship is far too complex to be describable in single statements. It seems as if we frequently manage to isolate some important ingredient of teaching or of learning and then make the cardinal mistake of elevating the part so that we begin to take it for the whole. The sentimental and mechanistic simplifications of the teacher's role that result when the more demanding pedagogical skills and expertise are ignored or denied provide sad examples.

II Although it is unlikely that anyone would dare to say aloud that mathematical education is *not* in need of humanising, so that we could easily take the case as read and get on to consider the more interesting question of how to do it, if we do choose to look at the reasons we find ourselves making analyses that may be transformable through reflection into statements about goals. If we notice that mathematics teaching often seems to generate fear and anxiety in children, we may want to humanise mathematics teaching so that these accompaniments are eliminated.

This in turn may suggest specific changes in our ways of teaching that might achieve this result.

But I don't want to pursue this question here for a mixture of reasons. I think it is too easy to become mesmerised by the multitudinous evidence of children's anxiety about mathematics into forgetting that some children appear to learn mathematics without experiencing it. I would also rather formulate goals and devise actions in terms of what I am *for* rather than what I am *against*. A preoccupation with the unfortunate consequences of some mathematics teaching may not help improve the situation because it may distract us from looking for preventive rather than curative measures. And I suppose I ought to be candid enough to admit that this question does not move me very much.

We can, of course, look for reasons to humanise mathematical education in the general social conditions of education and not in any special characteristics of the subject. As a small gesture to the value of taking this larger view, let me choose three arguments that seem to have particular relevance at this time.

(1) The knowledge-explosion and the huge technological advances of the post-war years have brought us all to the realisation that no one can possibly know all that it might be good to know, and cannot possibly grasp all the implications of discoveries whose applications nevertheless need to be controlled. It is very hard to know the position of individual responsibility in a society that appears to stumble into most situations through ignorance or ill-will. The individual's sense of puniness and powerlessness now seems to have little connection with the sense of insignificance which he could once generate in himself by contemplating the infinite mystery of the heavens. Each one now has to define the boundaries of his own universe, or, to say it another way, to construct his own universe within the conditions of his own interests, needs and powers. Only the development of the self's awareness of itself and its world seems to hold any hope of ensuring that the universe one actually chooses is not too small or just the objectification of one's wishes and fantasies.

(2) We belong to a society which succeeds in alienating groups and fissuring individuals, which engenders violence and provokes mental disorders. A great deal of educational practice unfortunately seems to collude with these processes, yet it is probably true that schools exert more ameliorating influences than most institutions, and it is therefore possible to conceive that education could want to do much more to heal the divisions between and within people. Regrettably, its attempts so far—through, say, the provision of social studies, counselling, subject integration and the acceptance of a degree of student choice—emerge as essentially trivial solutions incapable of substantially shifting the identification of schools with a particular form of society. The humanisation of education would require a far greater concentration on the steps that schools can take to develop the characteristics and exercise the powers of children and to show them that they can transcend their present world if they want to.

(3) The humanisation of education should also take care of the need to educate democratically for individuality. Schooling can no longer mould children into an acceptable form, or even show children an acceptable model. Perhaps it never could; but it has undoubtedly tried. There is no mould or model now since we cannot possibly find one which enough people would accept. The "individualisation" of education is now an absolute necessity, but should not be confused with a trivial innovative technique for organising classes which does not, in fact, achieve what its self-chosen label claims.

These three issues bring us up against a number of social, economic, political and ideological problems. And yet the aspect I choose to think about focuses on what can be done, however little it may be, by teachers in their classrooms in whatever circumstances they find themselves. The social circumstances need changing, for sure, but it is still the classroom that can determine the impact of education upon children and that cannot wait until the whole world is straight.

III But how can we humanise mathematical education, since this is the field of our own responsibility and the place where each of us can start? I choose three ways for mention here, conscious that the first two have already been considered and worked on by *ATM* members for some time.

(1) The substitution of the goal of facilitating children's mathematical activity for the goal of passing on mathematical knowledge is a substantial step in demonstrating that mathematics is a human activity. Activity is personal whereas knowledge often seems impersonal; activity is dynamic whereas knowledge is frequently inert; activity implies involvement in one's own learning rather than passive acceptance of someone else's.

But this is not any longer a new message and perhaps we need to consider how to avoid the danger that mathematical activity becomes a label for something too diffused and generalised, a way of learning in which almost anything goes. It may be another step, if only a relatively small one, to substitute for the encouragement of mathematical activity an education which zeroes in on mathematisation (an ugly word, but no matter). The shift of emphasis can take us even further away from an exclusive reliance on external criteria of quality derived from the mathematics of the past. Even though the aim to promote mathematical activity was designed to stress the importance of the "process" over the "product", we have tended to reassure ourselves that what we were encouraging *was* actually mathematical activity by making sure that the product was recognisably familiar mathematics. So, in a way, the nature of the product still dominates our judgments. On the other hand, the word "mathematisation" is a label for the process itself and provided we can back up our use of it by a strong rational conviction that children have the necessary functionings to be able to mathematise, we may be liberated from the tyranny of judging mathematics only by looking at what it has produced in the past.

The products of mathematisation are not, in fact, necessarily mathematics at all since the power to mathematise exists independently of mathematics as we know it, although it produced the mathematics we know. It is not far-fetched to see mathematisation behind many everyday activities —crossing a road against heavy traffic, driving to an unfamiliar destination, writing a word one has heard pronounced for the first time, running to intercept a ball, and so on.

In a crude attempt to make explicit the nature of mathematisation, I would include the following ingredients: the ability to perceive relationships, to idealise them into purely mental material, and to operate on them mentally to produce new relationships. It is the capacity to internalise, or to virtualise, actions or perceptions so as to ask oneself the question, "What would happen if . . . ?"; the ability to make transformations—from actions to perceptions, from perceptions to images, from images to concepts, as well as within each category—to alter frames of reference, to refocus on neglected attributes of a situation, to recast problems; the capacity to coordinate and contrast the real and the ideal and to synthesise the systems of perception, imagery, language and symbolism. When these functionings are applied to pure relationships, detached from specific exemplars, the products will then be mathematics.

For a presentation of one way to generate games and activities for children that engage their powers of mathematisation in order to get them to produce elementary mathematics of a familiar sort, I refer readers to Caleb Gattegno's *The Common Sense of Teaching Mathematics*.

(2) The body of mathematics as we know it is an accumulation from the work of many many people. It may appear to us as an imposing edifice with an objectivity and presence that we must respect even though we know we have glimpsed only a tiny portion of it. The vision of the edifice may well have inspired some people to become mathematicians in order to work in it; it has undoubtedly threatened and turned off many more who have seen it as impersonal, esoteric and inaccessible.

Perhaps it helps a little to know that it is not much like an edifice at all, although it contains some masterly constructions. It can be more realistically pictured as a rabbit warren with a few busy and well-lit chambers, a redundancy of passages, many deserted sections, and with even the most daring excavations extending only a little way into the claimable territory.

It would humanise mathematical education if we could present this accretion of results as it is, warts and all, so that learners might gain a sense of a human activity with all its admirable and foolish qualities. What questions were mathematicians working on when they produced such-and-such mathematics? Why has some work been preserved and other work lost? Could they always see the significance of their own achievements? What were the great technical and conceptual breakthroughs?

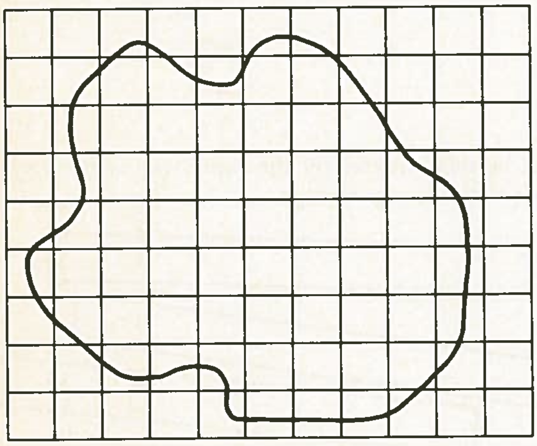
None of this can be a matter of direct experience for any of us now, even though we may be taken with certain analogies to our own mathematical experience. These questions are concerned with an enlargement of our experience and our understanding through vicariously sharing the experience and understanding of others. We read books and watch films and television for similar purposes and it is possible that these media can bring the mathematics of the past to us too. Of course we already use them to some extent, but I tend to think that what we are talking about here will not be achieved until we take for our models the novel and the feature film rather than the textbook and the documentary.

(3) The third way of humanising mathematical education, and the one which seems to require some special attention, is to utilise the lessons of awareness in order to educate children's awareness through the medium of mathematics. Awareness is a characteristically human quality, and awareness of one's awareness is possibly the most human state of all. The principle that "only awareness is educable" was first enunciated by Caleb Gattegno. See, for instance, *What we owe Children* (Routledge, 1971). I am deeply indebted to this insight and to many others that he has made available to me.

Like most important words, which are important chiefly on account of their richness and flexibility, "awareness" is difficult to define and I will not even try. I will assume that some resonance is immediately set up in readers and that more may become available as I go on using it.

Mathematics yields up information about the awarenesses it requires more readily than many other subjects since it chooses to make some of them explicit in the form of definitions. A definition is a formulation of an awareness, chosen for preservation in this way because it meets certain criteria of significance—that it emphasises attributes that might otherwise be overlooked or ignored, that it can be employed in a variety of situations, that it suggests possibilities of development and has a future. Not all mathematical awarenesses are captured in definitions, but if we want to find the necessary mathematical awarenesses in some area we are concerned with, definitions at least give us a place to look.

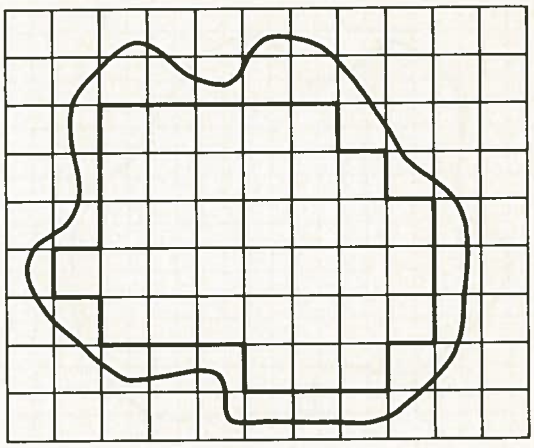
As an example, let us consider that familiar activity of primary school classrooms, finding the area of an irregular plane shape, like a leaf, say. It is usual to recommend the children to place the leaf on a printed square grid and "count squares".



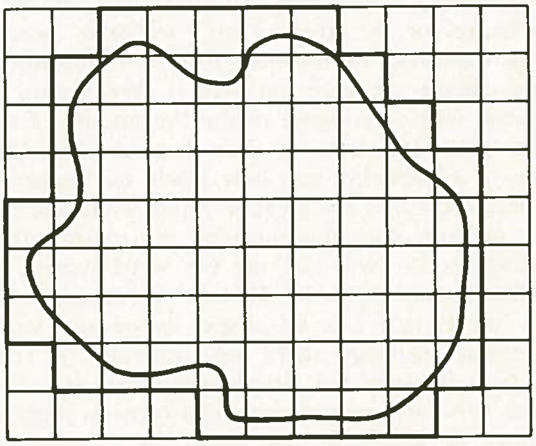
We usually take a ruler to the object we want to measure, but here we have brought the object to the measuring instrument. When we measure with a ruler we know we must place it carefully in a certain relation to the object, but here we do not seem to mind exactly where the leaf is placed on the grid.

When we have counted the "whole" squares contained inside the perimeter of the leaf, we have the problem of what to do with the odd bits. There are customary rules-of-thumb—say, counting 1 for each part-square which is bigger than a half-square and ignoring the others, or counting each part-square inside the perimeter as if it were a half-square. (Do these two rules give the same answer?) We notice that both of these stop the process of measuring instantly and make it impossible to consider improving the answers. Perhaps we should do more with the awareness that we are engaged in approximating (as in all measurement) and let the question of how to make the approximation "better" come to the surface.

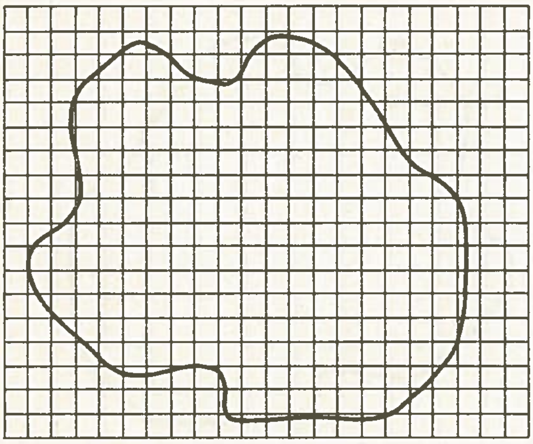
If we go back to the point where we have only counted the whole squares, we can import the awareness that we have found a "lower bound" to the area we want, i.e. a measure which is definitely less than the one we are supposedly after.



(Perhaps those readers who already know the term "lower bound" will see how this can function as a stored awareness that we can now put to use.) But if a lower bound, why not an upper bound? By counting all the squares which are entirely or partially within the perimeter, we get a measure which is definitely more than we want and we can be quite precise in saying that the leaf area is between two known areas.



These measures may well be quite far apart and we can now consider if it is possible to get other bounds that are closer together since this will clearly improve the information about the area of our leaf. Refining the measuring unit is a possibility (an awareness we have already developed in our experience of measuring lengths) and we can achieve this here by subdividing the grid so that it gives a finer mesh which will enable us to "follow" the perimeter more accurately. Although there is a practical limit to the extent we can subdivide, a limit reached pretty quickly, the action of doing it once can generate in us the awareness that it could be repeated at the virtual level (i.e. in the imagination) as many times as we want since there is no new ingredient entering the situation.



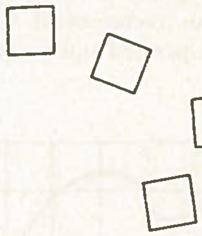
But we also require the leaf to have *an* area, i.e. for us to have some confidence that (i) it doesn't matter who does the measuring, (ii) it doesn't make any difference how the leaf is placed on the grid in the first place, and (iii) it doesn't matter what unit is chosen or what the precise steps in the subdivision are.

We reach a point, in fact, where our exploration has turned up a number of important awarenesses which we can now incorporate or take care of by constructing a definition. In effect this will be a definition of the area of an "irregular" shape which acquires its meaning from the constituent awarenesses we have put into it. We began, of course, with some sense of the "meaning" of the area of the leaf, but it is clear that the accumulation of awarenesses has now made our meaning more precise and has given us a tool which we can use to carry over this meaning into many other situations. We will still use the word "area" in different contexts with different meanings since the word itself has no single universally valid meaning (although there will naturally be connections between the alternative meanings). The analysis of meanings in terms of awareness suggests a way in which we can settle the relationship between different meanings.

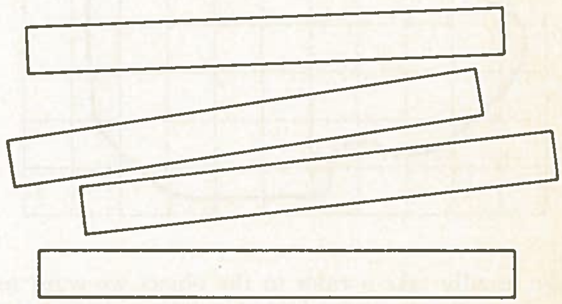
Let us consider another example to show how it is possible to display the awarenesses concealed in a simple situation. The example concerns multiples of 10, a little piece of elementary mathematics that gains its importance only because of the structure of our numeration system.

We may get cross if someone says that all that is required is to know that to form a multiple of 10 one must add a zero to the numeral representing the multiple. But apart from a tendentious use of the word "add" which can easily be avoided, what is wrong with this? Is it not what we do when we write a multiple of 10, not thinking about it, just doing it? At the level of functioning with notation, nobody needs to know any more.

But this rule doesn't take account of the different awarenesses that are required to operate at the level of speech or at the level of action. At the level of speech the awareness is that, except for a few irregularities, we form a multiple of 10 by making the sound "ty" immediately after speaking the numeral. Finally, at the level of action (and thinking in terms of coloured rods, although similar awareness can be reached in other terms), we can be aware that many of the things we can do with white rods—count them, group them, add them, etc.



can be done in exactly the same way with orange rods without any change.



That is, some things we know about "ones" can be immediately transferred to "tens".

The composition of these awarenesses really gives anyone all that he needs to know to function with multiples of 10. And if anyone raises the question of "understanding", I am prepared to maintain stoutly that this is all he needs to understand about the matter too.

Awarenesses are of all types and of all kinds of specificity. Some can only act as quite general backgrounds against which we can set the more particular awarenesses that we shall need. For example, we may be aware that the content of geometry is images whereas the content of algebra is mental dynamics; or that geometry belongs in space but algebra belongs in time. These awarenesses act as guides, giving us an orientation in our search for other awarenesses.

Some awarenesses, although quite general in their coverage, are more technical in nature. Although it is usual to pick out generalisation as a highly significant feature of mathematical activity, it is possible to become aware that it is far less significant than the role of specialisation or particularisation. Indeed it is the case that generalisation virtually looks after itself since it is implied as soon as mathematical discoveries are expressed in words or signs. Words and signs, except in a few cases, apply to classes of referrents and carry a built-in generality automatically. But in order to find what is *worth* saying we need to attend to the special features of situations, to the attributes which might go unnoticed. When we are pursuing some mathematical investigation we can become aware that all the forward movement is really generated in the moments when we decide to take some particularity into account. The rest, the generalising, is marking time, or patiently pursuing the consequences of a decision, or just consolidating. (It is the definitions that give momentum to geometry, not the theorems.)

Awarenesses often arrive unsought, in the middle of some other concern. But we can certainly look for them and open ourselves to them, especially by disciplining study so that we look for them where they are. We may want to consider afresh, for instance, the teaching of geometry and we might prepare ourselves by endeavouring to answer some particular questions.

What do children already know before we try to teach them geometry that we could use?
What appropriate functionings or powers do children bring with them?
Given that children already have relevant experiences and the capacities to work on them, what special structurings of their experience will lead to geometry?

Again, we may ask similar questions about the calculus. Another kind of question will also come up here. Calculus books are usually written at a level of rigour which satisfies the writer, whereas the wholesale employment of intuitive approaches in a pre-O-level course may allow the students to determine the level of rigour. But maybe the importance of rigour is not the search for the right level but the awareness that it is relative and that one can have additional control over the mathematics if one knows that the level of rigour can be raised, with certain consequences, or relaxed, and produce others. A calculus course which acknowledges the role of awareness will consciously include situations which force consideration of the effects of different decisions about the level of rigour.

IV Let me now throw away a few concluding and rather miscellaneous remarks.

Educationists are fond of talking of the primacy of experience and enjoy making statements about the teacher's responsibility to provide children with experiences. No doubt experiences are a necessary ingredient of learning, but they are by no means sufficient. We can all readily find examples in our own lives (and no doubt even more readily in the lives of others) which show experience sinking without trace. It is distinctly possible to experience a lot and learn almost nothing. But awareness is the act of attention that preserves the significant parts of experience, that pegs and holds them in the self so that they are available for future use.

Awareness is very nearly the same quality as insight, that subtle English word that hints at the possibility of "seeing into" by means of one's "inner sight".

Awareness has some of the characteristics of intuition but cannot be identified with it. Indeed, awareness can transcend intuition and reach where the latter cannot. Cantor provided us with data for the awareness that there is a hierarchy of infinities, a conception that intuition cannot possibly reach. Dedekind clarified the notion of continuity, which slips as elusively away from intuition as from the reason. (What are the great mathematicians for if not to enlarge our awareness with their own? Any competent mathematical hack can subsequently knock out the theorems.)

Awarenesses do not have the status of hypotheses or theories; they are knowings. We do not doubt an awareness for we know it as well as we know how to activate the muscles in our fingers or how to make a mental image of a place we know well.

Perhaps we are uncomfortable with the words "knowing" and "knowledge" and would rather avoid using them since they seem to commit us to a view of permanence and objectivity that we cannot feel we are entitled to hold. And yet the "knowing" of an awareness is not at all like a belief or the affirmation of a highly probable statement. The attempt by some philosophers to find a probabilistic domain which can accommodate guesses, beliefs and knowings on a continuum is entirely false to the existential experience that all three are absolutely distinct human functionings.

We can "know" and we can know that we know. It does not destroy the meaning of the word if we acknowledge that all knowing is necessarily both personal and relative; these qualifications, in fact, seem to me to rehabilitate the word and give it a power and relevance that make it indispensable.

Finally, awareness is a tool. It not only provides us with markers along the paths of our growth in skill, in knowledge, and in understanding, but it reveals to us the challenges, the questions and the problems that remain to be worked on. The education of awareness may appear to be an oversimple answer to the huge problem of humanising mathematical education, yet it is indeed the only answer that is capable of handling the complex challenge of providing an education that knows where it is, and shows the children where they are, and yet respects everyone's right to be educated independently of theories, ideologies and fashions and the inhuman demand that one should be content to be subservient and conformist.