THE CONCEPTS OF MEASUREMENT

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The publication in this country of a translation of *La Géométrie Spontanée de l'Enfant* by Piaget, Inhelder and Szeminska[1] is, as always with Piaget's work, an event of some importance for mathematics teachers. Many teachers, particularly at primary school level, will want to read this book and consider the implications of it for their teaching.

This volume is concerned with the development in the child of the concept of measurement. An earlier study ('The Child's Conception of Space,' by Piaget and Inhelder[2]) traced the evolution of projective and Euclidean notions of geometry from a more elementary intuitive perception of the topological properties of space. The present book continues the story by examining children's behaviour in situations in which the metrical properties of space need to be handled with precision. It shows how children pass from an egocentric unstructured perception of relationships of length and distance to the formation of concepts of measurement which enable them to devise their own methods of comparing lengths, angles, areas and volumes.

Euclidean geometry is the study of those properties of space which arise from the group of transformations which preserve the distance between any two points of the space. More simply, it is the geometry of 'rigid body motions'[3]. It is in this sense that it can be called the geometry of the real world for it corresponds to our experience of a space in which things are not altered in shape by being moved. (Contrary to what we might expect, children below the age of about 7 do not have this experience and are quite prepared to believe that objects change their size when they are moved about. The vital concept of conservation, as in the case of number, is surprisingly slow to form). Measurement is possible in this geometry because distances are conserved and the action of measuring involves an intuitive appreciation of the transformation group. The child must acquire, paradoxically, a qualitative understanding of measurement. It is this with which Piaget is mainly concerned.

Although the whole aim of the book is to trace the growth in the concept of measurement, in none of the early experiments is a measuring instrument provided. As Piaget says, there is little point in studying how children use a ready-made

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1 'The Child's Conception of Geometry': Routledge and Kegan Paul, 1960.

2 Routledge and Kegan Paul, 1956.
See also the article by P. C. Dodwell in 'Mathematics Teaching' No. 9.

3 It is possible to regard the Euclidean transformations as motions—rotations and parallel translations —or as correspondences of a certain kind between pairs of points. The former approach appeals more to the intuition and is adopted here; the latter is more appropriate in a systematic deductive study.

foot-rule until they have attempted the job of making one for themselves. One of the experiments in the first part of the book shows very clearly the gradual progress towards an awareness of all that is involved. A tower of wooden blocks standing on a table is shown to the children. They are given more than enough blocks, a supply of strips of paper and sticks, and asked to make a tower on another table the same height as the first. When they have finished building the tower they are asked if the two are, in fact, equal in height. At stage I[4] the only means of comparison of the two towers is visual transfer. The children 'look and see' if they are the same. At the beginning of stage II they use manual transfer in which they try to make visual estimation easier by bringing the two models close together. At the end of stage II they have reached the stage of body transfer when they use their bodies, by shoulder-height or span of arms, to mediate between the two towers. Now they are on the threshold of finding out that they can compare most easily by using a third object which is equal in length to the two towers. This they reach at the beginning of stage III and Piaget says their thought is now operational because they have grasped that if a = b and b = c then a = c. But they are still short of the full concept of measurement as unit iteration. Children in the early part of stage III can only use a stick for comparison if it is either equal to the two towers or longer (in which case they mark it off with a finger). But they cannot yet use a shorter stick effectively. This ability to use a unit appears at the end of stage III and they now have an understanding of linear measurement. It involves knowing that length is conserved — the stick used as unit does not change in length when it is moved, nor the tower when it is being measured—and that lengths can be subdivided on the one hand or composed additively on the other—the movement of the unit as it is stepped up the tower implies the existence of fixed reference points taken up by the extremities of the unit stick as it is moved.

This knowledge that changes of position can be related to fixed reference points is central to the whole notion of measurement (and to the construction of a Euclidean space). The many statements in the book about the need to construct a system of coordinate axes—which seem confusing if we think only of graph paper—imply just this. A small child is unable to make a correct representation of relationships in space because his view is essentially subjective and egocentric, deriving from his own actions. For example, if he is asked to show in a sandpit the positions of his home and school and other landmarks, he is capable of considerable distortion of distances and orientations as he cannot coordinate successfully the items of his experience and is unable, in general, to relate more than two things together at a time. The journey from school to home will be short because he traces it so often, whereas the buildings which do not concern him are put a long way from those that

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4 Piaget classifies the growth of understanding in four categories which approximately correspond to a division by age-groups. Stage I ('pre-operational' thought) lasts on the average from about 2 to 4½ years, Stage II ('intuitive' thought) from 4½ to 7, Stage III (the level of 'concrete operations') from 7 to 11, and Stage IV (level of 'formal operations') from about 11 onwards. In this study he further subdivides the early stages into two sub-stages, IA and IB, etc., where necessary. [The ages associated with each stage refer, of course, to the children with whom Piaget is usually dealing. The stages may occur at earlier or later ages with children in different environments. —EDITOR]

he uses. It is not until he realises that he must take certain landmarks as fixed reference points and not rely upon himself as a reference point (because he is not fixed in position) that he progresses to being able to represent positions in space with fair accuracy.

Further experiments examine in more detail the way in which notions of conservation and change of position are gradually clarified. When a screen or other obstacle is placed between two fixed toy trees standing on a table, the young children of Stages I and IIA believe that the trees become closer together. They are unable to appreciate at this age that the distance between the objects does not change because they distinguish 'filled' from 'unfilled' space. Only later do they gain a view of space as essentially homogeneous. And only at Stage IIB or IIIA do they become aware of the reversibility of a distance; that X is as far from Y as Y is from X. When asked to judge between the lengths of a straight stick and a snake of plasticine whose ends coincide with the ends of the stick, the young children assert that they are equal, even when they have traced the two lengths with their finger. Two straight sticks admitted to being equal when they are held side by side become unequal when they are staggered in position. And so on.

Put briefly like this it all seems hard to take, but it is illuminating to read the verbatim comments of the children made during the experiments as they show much more clearly than this bald summary can do the nature of their thinking. Their reactions then do not seem unexpected or ridiculous but essentially truthful to their experience—of objects, words, situations—at the time of the experiment. Complete insight when it is achieved is expressed quite unambiguously in language natural to the children. Although we cannot see their minds working, their actions and their statements together are pretty good indications of the stage their thinking has reached.

In the course of an experiment in which two identical strips of paper are cut into segments and arranged in two assymetric L-shapes, a child (aged 7 years 2 months) is asked:

"Is it the same either way?" (for two insects walking along the paths) (Hesitates) "It's the same (re-arranging the two segments as straight lines) because you cut them."

"And like this?" (two right angles made of different segments).

"That's a longer road and that one will have less far to walk."

"Why?"

"Oh no; that one has has a little corner missing; it would be shorter if it weren't like that. If they both walk together it'll be the same way for both."

"Why?"

"Because you cut bits of paper the same size."

"*And like this?*" (*increasing the number of angles and the differences in the lengths of the segments*).

"*Here it's longer because it bends . . . . Oh, no it's the same because they were the same to begin with.*"

Here we can see the attainment of the conservation principle which the child verifies, first actually and later virtually, by making the direct comparison between the initial and final arrangements of the strips of paper. And it is also clear that, at his present transitional stage, his visual perception very nearly inhibits his recognition of the equality. Children at a later stage have no such hesitations and are not put off by re-arrangements however distracting.

The experimenters use a very similar situation to investigate more fully unit iteration which is the climax to the whole process of understanding linear measurement. The pairs of paper strips are cut into segments whose lengths are multiples of 3 cm. and stuck onto cards in various arrangments. When the child has been asked if the two paths are equal or not, he is asked to verify his answer with strips of card 3 cm., 6 cm., or 9 cm. in length, the investigator giving the clue to the procedure by using the 3 cm. card to step off distances along one of the paths. In spite of this direction, which might be thought to give so much of the game away that there would be nothing to report, it is found that remarkably little acceleration in the formation of the concept takes place. Some of the children of Stages I and IIA apply the measuring unit to only one of the lines; some measure both lines with the same unit, obtain the same number of steps along each and still assert the inequality of the lines! Many of them fail to mark at all accurately the starting and finishing positions of the unit so that their steps overlap or leave gaps in between. Others again use uncoordinated mixtures of the available units. At Stage IIB, most of the testees are aware of the need to fix reference points and they understand the transitive nature of equality. But their understanding is not yet fully coordinated as they are quite capable of using different units unsystematically, or even of thinking that the two paths can be equal measured with one unit but unequal measured with another.

At Stage III, the various facets have all been successfully coordinated. The idea of subdivision of a line so that it can be reckoned in terms of a single unit-length complements and fuses with the idea of the changes of position of the applied unit being related to fixed reference points. The coordination of these ideas depends on the knowledge that lengths are conserved which is itself the culmination of transitive operational experience. Piaget maintains that the acquisition of the conservation concept precedes the full understanding of metrical measurement by about a year, on the average. He points to the similarity between the growth of the concept of number and of measurement but says that because quantification in measurement depends upon arbitrary subdivision of continuous wholes the idea of a unit comes at the end of qualitative understanding instead of preceding it.

All this may seem a good deal like obscuring the simple with a lot of words. It is certainly true that to an adult these elementary notions seem trivial but this is, perhaps, only because they are so fundamental to our ways of thinking, so much a part of our approach to spatial experience, that we are unconscious that they had, at some stage, to be aquired. Most of the time we are as unconscious of them as of many other things that we have learnt in the course of growing-up. What is interesting is that the full development of understanding in the child, if Piaget's theories are correct, corresponds so closely to the understanding which a mature mathematician has when he analyses the bases of his concepts from his more advanced standpoint. When we really stop to 'think about our thinking', to examine our fundamental ideas, we find ingredients very similar to those the child has to select and coordinate in order to arrive at his intuitive understanding. This exonerates such investigations as these from the charge of triviality. (It does not, of course, prove them right, necessarily). That the analysis may not be *useful* is another point. Does it help the teacher to know the various components that go to the making of a particular concept? I think the answer must be 'Yes'. From a broad view, nothing that helps us to understand how learning takes place can we afford to ignore. This is, after all, what we are concerned with and our teaching is only a means to that end. And in our particular subject it may well be that if we knew more clearly what was involved in the concepts that we want our pupils to acquire, we would be better able to put before them those situations which convey the 'raw material' of experience which they need.

We might also reconsider, in the light of suitable investigations, whether we make enough allowances for the fact that certain concepts do not seem to be reached until certain developmental stages. In this book, for example, Piaget suggests that children are not able to understand the calculation of rectangular areas and volumes until Stage IV (i.e. at an age of about 11). Quite a number of children are taught the rules for calculation in their primary school. If it is true that the concept cannot be mastered before they reach the secondary stage then it would certainly be very much better if this were not done. It is well-known that for many children (some of them not so young, either) area *is* length times breadth. But this is quite clearly not the result of them having a well-defined concept of what area is or how it should be measured.[5]

The development of the area concept follows very closely that of linear measurement. At Stage IIA children can cover an area with unit squares and count them even though they may have no clear idea of why they should do this; another area covered with less squares will not necessarily be thought of as having a smaller area. Children at level IIIA have begun to grasp the conservation of an area within a perimeter but do not associate this with conservation of complementary areas. In an experiment in which a 'potato patch' in a 'field' is cut up and rearranged, the recognition of the constancy of the area of the patch does not, at this stage, imply the constancy of the area of the rest of the field. Only at level IIIB is this fully operational understanding achieved. At this stage, too, the children can measure

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5 See the article by G. P. Beaumont in 'Mathematics Teaching' No. 12.

by unit iteration. That is, they can use a single unit square to measure a rectangular area by moving it systematically within the perimeter. But it is not until Stage IV that mathematical multiplication is called upon to find the area of a rectangle. There is apparently a real difficulty in using two *linear* measurements to reckon an *areal* measure, in spite of the fact that all the information would seem to be available. It is only at this level that they can even begin to understand what is implied in finding a square having twice the area of a given square—a situation which forces them to think in terms of the measures of the sides. (It was not expected that the children even at this stage would necessarily be able to compute the appropriate length accurately. They merely had to show an awareness of the real nature of the problem).

Similarly the development of the concept of volume passes through successive stages until the full physical concept is reached at Stage IV. Before this (at level IIIB) conservation of 'internal' volume is assured but strangely this does not imply constancy in relation to surroundings. A ball of clay or a collection of wooden blocks will be agreed to remain constant in quantity and yet be capable of displacing different amounts of liquid in different transformations!

Even though some of these results—by no means all have been mentioned—may seem to deserve a sceptical if not incredulous reaction, it is evident that matters of great importance to teachers—and especially primary school teachers—are being dealt with here, and that the book should be read by some of those concerned with mathematics teaching at this level. There is material for plenty of further experiment with children: to confirm Piaget's conclusions; perhaps to refute them; perhaps to extend them. If they are substantially true they must influence what we do in the classroom.

As one reaches the end of the book, admiration is mixed with irritation. Does the language (I do not blame the translator) have to be so obscure? Some sentences, even when read attentively many times, still refuse to give up their secrets. The commentary is often repetitive whilst failing to clarify and too often gives the impression that it is forcing the observations into the expected, necessary pattern ordained by Piaget's theories. No details, other than age, are given for any of the subjects of the experiments. We would like to know more about them. Are they typical, normal children? Are all their recorded remarks, quoted in such detail, completely unselected? One fears not.

These criticisms made of almost any other report of a psychological investigation would be pretty well sufficient to damn it. But in spite of the book's faults and the determined patience required in reading it, the admiration at the end outweighs the irritation. The profound simplicity of the experiments and the insight which guides the analysis of the responses carry their own conviction. And it is Piaget's genius that he has perfected a method of psychological investigation which is wholly suited to the task of understanding which he sets himself and which has, in his hands, a subtlety generally lacking in other methods.

The essence of Piaget's experimental method is the observation of children's spontaneous efforts at problem-solving. Since he is not concerned with mathematical achievement as it is influenced by teaching, he tries to ensure that the experiments do not need knowledge acquired in the educational process. It is central to his whole approach that the basic concepts he is examining are arrived at spontaneously at some developmental stage (depending on the quality and complexity of the concept). At no time has he suggested that these concepts could (or should) be taught in advance of the natural stage of attainment and he does not consider how far the experimental situations he devises may act as learning situations in which the concepts crystallise.

It is clearly of the utmost importance that investigations should be directed to shedding light on these two related questions. Some experiments that have taken place do, in fact, suggest that the kind of situations which Piaget devises for his children, can be used to precipitate or at least accelerate the formation of concepts, but great care is obviously needed in work of this sort to avoid the danger of a circular argument in which experiments merely validate themselves. Perhaps it is Piaget's realisation of this which has led him to omit such considerations.

In reviewing another book by Piaget ('The Growth of Logical Thinking'), J. S. Bruner[6] points to a more serious omission—that Piaget fails to take into account the motives which actuate children in their thinking. This lack of consideration of strategy and goal-direction creates (or does not clarify) one of the most mysterious features of Piaget's view of thinking: the equilibrium—to use his own word—of the stages of concrete operations and of formal operations. As Bruner shows, it is necessary to examine why the former stage, if in a state of equilibrium, ever yields place to the next and in what way the modes of thinking of an adult differ from those of a child of 12 who has reached the level of formal operations. Piaget's own explanations of these shifts are not convincing. No doubt other investigators, inspired by his work but not afraid to modify his theories, will eventually answer this question.

In the meantime, we can be grateful for yet another fascinating book which dazzles us with the brilliance of its experiments and throws so much light on the complexities of fundamental notions. It demonstrates once again the power of Piaget's methods in illuminating the processs and progress of mathematical thinking. Instead of taking elementary mathematical ideas for granted, as trivial, as uninteresting, he patiently and lovingly delineates the steps by which they are reached. Children's failures are shown to be the fumbling first moves in the right direction and the successes are their inevitable reward in due course. There is no stupidity in Piaget's world. There is only immaturity.

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[6] British Journal of Psychology, November, 1959.