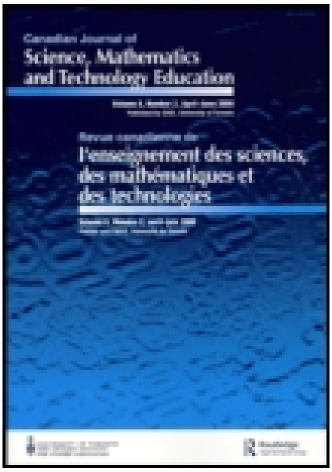

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The Importance of Teachers' Mathematical Awareness for In-the-Moment Pedagogy

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The Importance of Teachers' Mathematical Awareness for In-the-Moment Pedagogy

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Abstract: We propose that what teachers can say or write about their views of mathematics and about a range of pedagogic strategies and didactic tactics that they use is of minor importance compared to what comes to mind moment by moment when they are planning or leading a lesson. Beginning with a quote from a teacher that provoked a focus on the role and nature of in-the-moment choices made in the midst of teaching, we offer some comments provoked in us. Following a concise history of attempts to articulate 'mathematics needed for teaching', three further brief accounts of incidents led us to propose that what really matters, in the words of Heidegger and others, is one's 'mathematical being', because it orients awareness and is the source and basis for choices, whether consciously or unconsciously made. The development of mathematical being is described using the discourse of awareness in the sense of Gattegno, making use of the discipline of noticing to educate awareness. These are then used to extend the analysis of prepositional suffixes for the gerund *knowing* begun by Ryle in order to focus on the development of 'connective tissue' between mathematical awareness and in-the-moment pedagogical choices of actions.

Résumé: À notre avis, ce que les enseignants peuvent dire ou écrire sur ce qu'ils pensent de certaines questions mathématiques et des différentes stratégies pédagogiques et didactiques qu'ils utilisent, a peu d'importance comparativement à ce qui leur vient à l'esprit sur le moment lorsqu'ils planifient ou lorsqu'ils donnent un cours. Partant du commentaire d'un enseignant qui nous a conduits à centrer notre attention sur le rôle et la nature des choix faits sur le moment pendant les cours, nous présentons nos propres commentaires. Après un bref compte rendu des tentatives précédentes d'expliciter « les connaissances mathématiques nécessaires pour enseigner », trois autres brefs comptes rendus d'incidents nous ont conduits à l'hypothèse suivante : ce qui compte réellement, comme le dit entre autres Heidegger, c'est « l'être mathématique », car il oriente la prise de conscience et est à la base des choix qui sont faits, qu'il s'agisse de choix conscients ou inconscients. Le développement de « l'être mathématique » est décrit en termes de prise de conscience au sens où l'utilise Gattegno, c'est-à-dire qu'il consiste à remarquer pour former la conscience. Suivant Ryle, nous procédons ensuite à une analyse des suffixes prépositionnels associés au gérondif « connaissant » pour tenter de mieux définir le développement des « tissus conjonctifs » qui lient la conscience mathématique et les choix pédagogiques spontanés.

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PROVOCATION

Account 1

> Why aren't they getting this? . . . Either my class isn't very smart, or I'm not doing this correctly!

That remark began the first after-school debriefing session of a month-long project designed to introduce teachers to a more in-the-moment manner of mathematics teaching than they were currently practicing. The teacher was speaking about her frustration at being unable to mimic an introductory folding-based fractions lesson that she and her colleagues had watched me (B.D.) teach earlier in the day. She had taken copious notes on her observations, recording questions, responses, insights, connections, points of emphasis, and so on, aiming at an as-true-as-possible reenactment of the lesson in her own classes.

Accounting for Account 1

The incident reported as Account 1 took place at the beginning of a month-long project in an underperforming middle school, which was intended to support a more responsive manner of mathematics teaching. But when this teacher (and all four of her participating colleagues, for that matter) attempted to repeat the teaching of fraction equivalence through folding and cutting (real and imaginary), she, like the others, was frustrated that her students did not generate the same insights that had arisen in the modeled lesson, and so she felt herself compelled to tell and explain, rather than listen and engage. What struck B.D. in the event, and J.M. on encountering this account, was the readily recognized common phenomenon of switching into telling and explaining when things are not going to plan, especially for teachers in the beginning stages of trying to "work differently."

The account is multilayered, because it articulates a teacher's observation together with some background information and an interpretation by the teachers' teacher (B.D.) of the teacher's attempts to "capture" the essentials of a model lesson. The notion of an as-true-as-possible reenactment is, of course, tempered by what the teacher was attuned to notice and by what the teachers' teacher was attuned to notice.

As the debriefing meeting unfolded, it became clear that all of the teacher-participants had more than sufficient disciplinary knowledge to follow the trajectory of the lesson, so the issue was not any lack of understanding of the mathematics. They also understood the pedagogy: the intentions behind building up the example space, drawing out thoughts, bouncing back ideas, challenging interpretations, etc. They simply seemed to lack the vital connective tissue between *mathematical awareness* and *in-the-moment pedagogy*. It is one thing to notice an absence of something from a learner but quite another thing to have a sensible pedagogical action come to mind when needed.

It might be tempting to suggest that this teacher's frustrations were borne out of a naïve expectation that events would unfold in the same way when different learners were involved, but that interpretation is unfair. With nearly 20 years' experience, she was well aware that offering students genuine opportunities to interpret and speculate would require her to adapt on the fly. Even so, she was not expecting her students' interests and contributions to be quite so different from those in the observed lesson, nor was she prepared for small differences amplifying into irreconcilable trajectories. To cut the story short, as she reported, her detailed "observation notes were useless 10 minutes into class." She did attempt to stay true to the intention of inviting students to question and invent but confessed that a good deal of the lesson was spent "telling [the students] what they were supposed to be getting out of this."

Provoked Questions

If it was not a naïve expectation, why did this teacher feel compelled to opt for telling and explaining rather than listening to and engaging with her students? In this article, we argue that this sort of commonly noted phenomenon is mostly a matter of teachers' disciplinary knowledge of mathematics. That is, though there may be associated issues with personal theories of learning, conceptions of the nature of mathematics, teaching styles, and so on, we argue that the major factor in teachers' capacities to engage flexibly and productively with their students—that is, to engage in what we are calling *in-the-moment pedagogy*—is their own mathematics. Not, we hasten to add, the fact of having passed mathematics exams but rather the scope and range of mathematical thinking, associated pedagogical strategies, and didactic tactics that are available to come to mind, in the moment.

There is something special about teachers' disciplinary knowledge of mathematics that enables (or disables) in-the-moment pedagogy. However, it is difficult to bring to articulation without turning "it" into something else. Put another way, "to express is necessarily to over-stress," and, consequently, to underplay possibly important aspects (Mason, 1994, p. 177). Preserving the complexity of interactions between teacher, student, and mathematics is difficult when cultural and institutional forces are in the direction of taxonomy and reductionism, treating individual aspects as if they could be isolated from the many others.

As we develop presently through a brief review of the research into teachers' mathematical awareness, it seems that the questions that have oriented most investigations have been sliding past what really enables in-the-moment pedagogy, and our aim in this article is to suggest alternative foci while preserving complexity.

A CONCISE HISTORY OF RESEARCH INTO TEACHERS' DISCIPLINARY KNOWLEDGE OF MATHEMATICS

One consistent feature of recent studies of teachers' mathematical understandings is that most focus on explicit knowledge that might be explicated, assessed, and certified. This emphasis is perhaps unsurprising. The practices of certifying and certificating educators fits well with Aristotelian notion of a master's degree as signaling a state in which one is able to teach others. Unfortunately, in some settings (notably, the U.K.), as master's degrees have come to be the standard for initial certification of teachers, they have also come to be associated with passing mathematics exams and writing academic papers about teaching and learning.

On the surface, the matter of teachers' mathematics knowledge for teaching (MKT[1]) seems fairly straightforward. Teaching something deliberately such as how to sail a boat, how to tie a shoelace, how to factor a quadratic, indeed almost *anything*, demands a knowledge of whatever is being taught at a level that surpasses the current knowledge of whoever is being taught.

However, insofar as mathematics teaching goes, this thinking is at least problematic and, experientially, deeply flawed. As Begle (1972, 1979) showed, there is little or no correlation between teachers' college credits in mathematics and the performance of their students. Research into teachers' disciplinary knowledge of mathematics began in earnest in the 1970s, and the work of Begle is most often cited as seminal to this branch of inquiry. His studies exemplified a conception of MKT that dominated research through the 1970s and 1980s, as academics grappled with the question:

> What *mathematics* do teachers need to know in order to teach mathematics?

In the decades following Begle, researchers continued to refine the construct of teachers' mathematical preparation, moving beyond a simple count of postsecondary courses to analysis of the grades the teachers achieved, the instructional approaches taken, and the actual content covered (see, e.g., Monk, 1994). Yet the results were, for the most part, unenlightening.

Even so, there was no apparent decline in the near-universal conviction that knowledge of advanced mathematics was a vital part of teacher preparation. The weak evidence base in support of this strong conviction was no doubt a principal motivator for a renewed surge of interest in the 1990s. By then, a new way of thinking about teachers' disciplinary knowledge had swept through the field of education: pedagogical content knowledge (PCK). Proposed by Shulman (1986, 1987), PCK was described as follows:

> . . . for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others. (Shulman, 1986, p. 9)

Shulman was expanding and refining John Dewey's perspective that the focus for a teacher is on "psychologising the subject matter"; that is, with how "the subject matter can become part of the experience of the student" (Dewey, 1902, p. 22).

PCK sparked a new interest in MKT within the English-speaking mathematics education research community, with attentions shifting from formal (course-based) content knowledge to the subtler elements that arise in teaching. Notably, this emphasis was already represented in a significant body of research in several other languages. In particular, it was closely aligned with the European notion of *didactiques/didactiks*, as exemplified in Freudenthal's (1983) lengthy and detailed exploration of didactic aspects of mathematical concepts.

During the 1990s, an increasing number of studies picked up on PCK as a means to reframe the mathematics-for-teaching question. Prominent among them were Ma's (1999) case-based contrasts of the considerable mathematical PCK of Chinese teachers to a comparative lack among American teachers. Ma (1999) used the term *profound understanding of fundamental mathematics* to refer to this knowledge. She explained:

> A teacher with [profound understanding of fundamental mathematics] is aware of the "simple but powerful" basic ideas of mathematics and tends to re-visit and reinforce them. He or she has a fundamental understanding of the whole elementary mathematics curriculum, thus is ready to exploit an opportunity to review concepts that students have previously studied or to lay the groundwork for a concept to be studied later. (p. 124)

Ma's (1999) study triggered a rethinking of the orienting question of MKT among researchers, and efforts to identify the specialized types of teachers' disciplinary knowledge dominated MKT research in the first decade of the 21st century. In that time, the orienting question for researchers shifted from, "What *mathematics* do teachers need to know in order to teach mathematics?" toward

> What *specialized mathematics* do teachers need to know in order to teach mathematics?[2]

It is important to emphasize that the earlier question was not lost in this revised formulation. The shifting focus was never intended to suggest that courses in formal mathematics do not matter but rather to signal other categories of know-how that up to that point had not been given the attention they merited.

A further refinement of the question came a few years later, when Ball and colleagues (e.g., Ball, Thames, & Phelps, 2008) suggested a new vantage point. They pointed out that the mathematical knowledge of teachers is not static and should be thought of as knowledge-in-action. In their view, mathematics teaching is a form of mathematical practice that includes

- interpretation and evaluation of students' work,
- correlation of students' mathematical results with the processes of their production, and
- construction of meaningful explanations, and assessment of curriculum materials, among other elements.

Ball and colleagues called for a practice-based theory of teachers' mathematics knowledge that focuses on the specific knowledge that teachers use in their daily work. The framing question for the study of MKT became

> What specialized mathematical knowledge *is entailed by the work* of teaching mathematics?

Once again, this formulation involved an elaboration of earlier foci and not a rejection.

In contrast to the second iteration of the MKT question, which turned attention toward *knowing more*, this third iteration also gestured toward *knowing differently*. We acknowledge this shift in emphasis to be an important one, but we cannot help but feel that the line of questioning continues to slide past the way one must be *with* mathematics, in relation *to* mathematics (Handa, 2011), in order to teach effectively moment by moment.

FURTHER PROVOCATIONS

Account 2

> I [J.M.] am preparing to give a seminar about an analysis of sets of exercises in a series of 19th-century algebra texts. I am aware of all sorts of possibilities coming to mind. The fact of the imminent audience seems to heighten my awareness of possibilities: themes that can be illustrated and connections that can be made. I make sure that my notes contain records of many of these thoughts.
>
> From experience I am aware, while preparing, that I won't be able to check in my notes during the seminar, despite the pervasive 'sense' that if I have got something in my notes then I will have access to it.

> In the seminar some technical questions are asked whose answers I am sure are in my notes, but my attempt to scan through them quickly in real time simply gets me in a muddle.

Accounting for Account 2

I know, or certainly have known, a great deal about the topic, about what came before and after the extracts I had chosen, and about how they related. But in the moment, many of these simply did not come to mind. In a more relaxed state they might have, but a certain tunnel vision in the seminar blocked out access. Despite being aware in the moment in the seminar, actions such as taking my time to look properly were simply not available to me (J.M.). What is striking is the direct parallel with students working on problems in mathematics, as the next incident illustrates.

Account 3

> I [J.M.] was working with an undergraduate in a one-to-one tutorial. The problem was to show that two integers differing by an even number have a product that is the difference of two integer squares. She had expressed any two such integers as n and $n + 2k$ and was looking at the product, expanded as $n^2 + 2kn$. After perhaps a minute of her gazing, I asked whether anything spoke to her. Eventually I asked her what the square of $a + b$ was, and she replied immediately and then realized that adding k^2 to her expression gave a perfect square, completing her reasoning.

Accounting for Account 3

I could see that she was gazing at the expression, but neither indirect nor direct questions revealed how she was attending to it. Whatever details she was discerning were not leading to a pertinent relationship as an instantiation of the property of the square of a binomial expression. Despite knowing very well the expansion of a binomial square, the missing k^2 did not come to mind when needed.

While attending lectures, reading through lecture notes and textbooks, "things" may seem clear enough, but when confronted with something fresh or novel, the constructs and techniques may not readily come to mind in a useable form. A fill-and-drill-based pedagogy might argue for "more practice"; however, it is not facility that is absent but having an appropriate action come to mind. Another explicit example was described in detail in Scataglini-Belghitar and Mason (2012), in which students did not think to use a technique based on a theorem presented in class, with examples, and following a tutorial on a problem of what, to the tutor, was of a similar type.

For us, these two events illustrate the fact that "having knowledge" is not a useful way of speaking, because possession is not an appropriate metaphor. Rather, "knowing" is not so much about *having* as it is about *doing*. What matters is what actions come to the surface, either resonated metaphorically or triggered metonymically, in the moment.

Instead of treating knowledge as a possession, an enactivist perspective treats knowing as synonymous with action (Varela, Thompson, & Rosch, 1991), so not acting signals either a choice not to act or, possibly, incomplete or disconnected knowledge. This resonates with the notion of *inert* knowledge introduced by Spencer (1878) and picked up by Whitehead (1932). It also fits with the search for conditions under which *transfer* would occur (Detterman & Sternberg, 1993) between familiar and the unfamiliar situations.

Account 4

> [Lena, teacher:] Last month, my Grade 3 class [age ∼9 years] was counting down from 95 by threes. When they got to 2, I thought they would stop. But the counting continued to -1, -4 and so on. After a while, they stopped and realized that the numbers were increasing [in absolute value]. Then a boy deep in thought blurted out a question, "Can zero be a negative number?"
>
> The whole class became alive and engaged in hypothesis, proof, different representations on the chart paper, convincing theories and a genuine pursuit of knowledge. It became a conversation about whether zero could be a negative number. The class was very engaged and I heard a lot of mathematical language and thinking.

Accounting for Account 4

At the time of this narrative, Lena was enrolled in a cohort-based master's program focused on MKT. In particular, much of her coursework dealt with the subtle complexity of teachers' disciplinary knowledge of mathematics. In the grander story of Lena's teaching life, this moment stands out because it actually surprised her. She found herself able to hear something she hadn't heard in her previous 20+ years of teaching or, in the language of the title, she'd found a link between her *mathematical awareness* and *in-the-moment pedagogy*. She knew the mathematical answer, but she was able to use this to inform her listening to what they were saying, to the ways in which they were displaying mathematical thinking, rather than listening for an expected or correct answer (Davis, 1996). She was therefore able to support her students in using and refining their own mathematical powers in order to make sense of, to construct meaning concerning the apparently familiar concept of negative number in addition to other mathematical ideas.

MAINTAINING THE COMPLEXITY OF EXPERIENCE

Few would argue with the statement that effective teachers are not simply repeaters or distributors of knowledge—the 'sage on the stage'—despite the desires and rhetoric of policy makers and institutions. Nor are teachers simply "guides on the side" (King, 1993). Similarly, few would quibble with the assertion that teachers can take a range of stances (Bruner, 1986) depending on their assessment of the affordances, constraints, and student attunements of the situation (Gibson, 1979; see also Greeno, 1994). A perhaps more contentious suggestion, however, is that to understand the lived experience of teachers it is necessary to probe more deeply into what knowing-in-the-moment can be like, in order to suggest how teachers can develop that knowing, without losing the essential complexity of the phenomenon. In fact, we suggest that it all has to do with 'being': teachers being mathematical with and in front of their students.

We use the term *being* in the sense of Heidegger (1927), who used a crossed-out version to indicate that being is not some "thing," some aspect that can be detected and measured. It is a portmanteau for 'being in the situation', sensitized and responsive to what emerges, informed by personal experience of mathematics, learning, and teaching. We suggest that it is that which enables someone to *be* fully mathematical with and in front of students. In order to do this, they need to work at developing their being through engaging in mathematical thinking for themselves and with other like-minded colleagues.

Aspects of being have been elaborated by a wide variety of authors. For example, Cuoco, Goldenberg, and Mark (1996) developed the discourse of *habits of mind* to try to capture the attitudes and dispositions of mathematical thinkers. Halmos (1985) and Krantz (2012), among others, have used autobiography. Schoenfeld (1985) focused on mathematical problem solving, and Mason, Burton, and Stacey (1982/2010) rearticulated advice given by Pólya (1962) presented from an experiential-reflective stance. Davis (1996) highlighted a different aspect of in the notion of 'hermeneutic listening' in order to foreground participatory and interpretive aspects of teaching mathematics. For example, he distinguished between teachers listening to what students may be trying to express and listening for what they as teachers want to hear from students. This distinction aligns with Rowland and Zazkis's (2013) discussion of the role of disciplinary knowledge in enabling teachers' sensitivity to learner contributions. Changing stance can alter what actions then come to mind in response. This is an example of developing a sensitivity to notice, of educating awareness so that a richer collection of possible actions come to mind, of developing one's mathematical awareness can best be elaborated using the discourse of *awarenesses* and *knowings* linked together with the notion of the *discipline of noticing*, as discussed in what follows.

AWARENESS

A teacher who is aware not only of the subject matter but of the pedagogical aspects of the subject matter is in a position to direct student attention to what really matters, to what choices are available and to what criteria might be applied, when working with students on exercises and worked examples. But what does it mean "to be aware of something"?

Using the term *awareness* in the sense of Gattegno (1970, 1987) to mean 'that which enables action', Mason (1998) proposed several levels of awareness of importance in educational contexts:

- *Awareness-in-action* applies to the ground level awareness that involves specific (in this case mathematical) actions such as one-to-one matching to enable counting.
- *Awareness-in-discipline*, which involves awareness of awareness-in-action, is what enables actions to be initiated and mathematical thinking undertaken, independently of direct instruction.
- *Awareness-in-counsel* involves awareness of awareness-in-discipline, so that learners attentions can be directed appropriately, their powers evoked, and relevant mathematical themes encountered or invoked. These layers of awareness could be extended to a further awareness of awareness-in-counsel which is required in order to work with colleagues on issues in teaching.

Note that in this usage, awareness is not the same as consciousness. For example, somatic functioning is enabled by subconscious awareness, and internalized acts that come instantly to mind—such as counting, complements to 10, or use of symbols to express generality for those who have gained this facility—are activated because of subconscious awareness.

With this sense of awareness, actions initiated in the moment come to mind because of awarenesses that, in Gattegno's terms, "have been educated." In other words, mathematical has been enriched. Indeed, Gattegno suggested that that is all that can be educated, in contrast to behavior, which is what can be trained (Mason, 1998; Young & Messum, 2011). So what are the mechanisms by which actions come to mind?

KNOWINGS

One way into this question is through expert/novice research, which has a long history in the mathematics education literature (see, e.g., Leinhardt, 1989; Schoenfeld & Herrmann, 1982) and a very deep history in psychology and related domains (e.g., Chase & Simon, 1973; Ericsson, Charness, Feltovich, & Hoffman, 2006). A key finding among these studies is that, within their domains of strength, experts are able to remember more, respond more quickly, and deal more effectively with unfamiliar material. They also make it look easy, which means that key choices may be overlooked by novices when observing. These findings align with problematizings of commonplace distinctions between deep/superficial and conceptual/procedural knowledge (cf. Star & Stylianides, 2013)—dyads that can mask much of the subtle complexity of the expert's knowledge.

More concisely, what distinguishes expert from novice behavior is the degree of complexity of awareness that can be sustained moment by moment and the richness of possible choices that come to mind. Teaching novices to be teachers involves not only working on relevant mathematics together but developing a disposition to work on mathematics for oneself, whether by oneself or with others, to develop mathematical being. It also involves becoming sensitized to the psychological and sociocultural aspects of being a learner and doer of mathematics and to what it is like to encounter epistemological and pedagogical obstacles (Bachelard, 1938/1980; Brousseau, 1983). To be effective and practical—or, in other words, to develop knowing to act in the moment—these two strands need to be integrated through intentional work along the lines suggested by the discipline of noticing, namely, imagining oneself acting appropriately in the future and using this to sharpen one's noticing in the moment, thus educating one's awareness.

In order to elaborate and probe more deeply, we need to look at types of knowing, in preference to knowledge per se. The noun *knowledge* carries with it a sense of static existence, one that is often discussed as though it could exist independent of a knower: either someone possesses it or they do not, and their not possessing neither alters nor denies its status as knowledge. The gerund *knowing*, in contrast, refuses a separation of knowledge and knower. Further, as T. S. Eliot (1935) developed in "Murder in the Cathedral," our usual state is one of "knowing and partly knowing." That sense of "there is more to it, but it is as yet inexpressible" is a common state when preparing a course or an individual lesson. It can even be present as a vague sense in the midst of the tunnel vision of acting in the moment under pressure, when flexibility and adaptability are at a premium because preparation has not envisioned what actually happens.

Ryle (1949) used the fact that *knowing* in English takes a number of prepositions to distinguish subtly different types of knowing, which we have augmented slightly:

- Knowing about some topic (but often with that sense that there is more to know)
- Knowing that something is the case
- Knowing why something is the case (and so being able to justify knowing that)
- Knowing what to do in some circumstances
- Knowing how to do something

- Knowing when to do something
- Knowing to act in the moment

'Knowing about' more or less subsumes all but the last, because they are all associated with past performance, and they are often tested by means of essays and dissertations. These give evidence of accumulating knowing that, knowing why, knowing what, and knowing how and then stringing it together into some sort of a narrative but in settings quite different from in-the-moment choices in a classroom. What matters in teaching is knowing to act; that is, having knowing how, perhaps informed by knowing why, come to mind.

Knowing in its various forms spans a spectrum from the tacit to the explicit. Biggs (1994) rehearsed Ryle's (1949) distinctions and then directed attention to what he considered to be a different way of considering knowledge, in five hierarchical levels following Bruner's (1986) spiral curriculum and resonant of van Hiele's levels:

- *Tacit*: manifested through doing without conscious awareness being able to account for the origin
- *Intuitive*: directly perceived or felt
- *Declarative*: description of how and why, expressed in some symbol system that is publicly understandable
- *Theoretical*: abstracted or generalized statements going beyond particular instances
- *Metatheoretical*: knowledge about the process of abstraction and theory building.

Tacit knowing (drawing on Polanyi, 1958) refers to that which enables and initiates actions and which has been automated and habituated, so that they comprise professional practice. These are Gattegno-type awarenesses lying below the surface of consciousness and corresponding to craft-knowing, informing actions without conscious thought. Intuitive knowing is also beyond or beneath articulation and comes from direct perception informed by experience. Attempts to articulate intuitive knowings are unlikely to be informative, because they have to do with the sensitivities, the mathematical and pedagogical being of the teacher, as well as the richness of accessible past experience. Declarative knowing corresponds to various aspects of knowing about that can be articulated and narrated but not necessarily performed, whereas theoretical knowing includes knowing why or at least having a narrative by which to justify or ground choices of actions. This is the root meaning of *responsible*, namely, 'to be able to justify if required'. Thus, the responsible teacher has a theoretical frame by means of which to justify his or her choices of actions. Finally, metatheoretical knowing takes knowing in a different direction, turning mathematical thinking itself into a subject of study. Biggs (1994) actually included two additional forms, *procedural* and *conditional*, which do not seem to fit into his hierarchy and, in fact, both of these can be experienced at each level.

In order for these various sets of distinctions to inform future action, it is necessary to sensitize oneself to phenomena to which they apply and to accumulate and become familiar with actions that can make use of such opportunities. This is the aim and purpose of the *discipline of noticing*.

THE DISCIPLINE OF NOTICING

William James (1890/1950, p. 628) noted that “a succession of feelings does not add up to a feeling of succession,” and the same applies to experience: a succession of experiences does not add up to an experience of that succession. In other words, it usually takes more than mere experience in order to learn effectively from experience. A scientific approach seeks logical, cause-and-effect chains of reasoning and action. However, human beings are not so much rational as “capable of reason” (Swift, 1726), an insight that presses attention toward the experienced, unvoiced, tacit substrate of mathematical work. Tacit and intuitive knowing to act in the moment are activated through two mechanisms: metonymic triggering and metaphoric resonance.

Metonymic triggering concerns associations arising from surface resemblances, often based in emotions. Metonymies are notoriously difficult to trace, being largely idiosyncratic and very rapid. They are responsible for the almost chaotic flow of conversation and for the coming to mind of unexpected or surprising connections. Metaphoric resonance concerns structural resemblances leading to analogical thinking and reasoning (English & Sharry, 1996; Rumelhart, 1989). This can take place well below the surface of consciousness—but it can also be carried out intentionally, developing metaphor into analogy and exploring consequences. Sometimes metonymies and metaphors can be traced or justified by a post hoc analysis of what ‘must have happened’, but it is difficult to trap the experience of the moment.

In the moment, people react out of habit and respond out of sensitivities and predispositions. Reactions come from the internalized, the automated, the habitual. Though it is sometimes possible to trace the origins, often it is not. Responses, on the other hand, are more considered. In-the-moment choices between actions are most likely to be reactions, unless the person is able to split his or her attention to attend to both the choice and the fact of the choice. The French expression *l’esprit d’escalier*, referring to that moment after an interaction when you wish you had thought of saying or doing something differently, points to knowing that emerges more slowly, often requiring quiet reflection in order to move from the retrospective to in-the-moment “spectivity.” Gaining access to this kind of knowing in the moment of activity is what awareness-in-discipline and awareness-in-counsel are pointing to.

The only strategy, the only action that human beings have access to in order to learn from experience is to become consciously aware of recent actions that proved fruitful and then to imagine themselves having this action come to mind in some similar situation in the future. It is the use of the power of mental imagery, the basis for planning, that creates conditions in which a prepared action comes to mind when it is appropriate. This is the basis and aim of the discipline of noticing (Mason, 2002).

KNOWING MORE . . . AND KNOWING MORE RICHLY

Given the tenor of the preceding, the phrase *knowing more* is too suggestive of accumulating yet more facts and theories. Though these can be an important component, what we are concerned with here is more a matter of *noticing more* than *knowing more*. That is, we are concerned with knowing more deeply and richly in the sense of having possible actions—mathematical, pedagogic, and didactic—come to mind when they are needed, whether when planning or in the midst of activity with students. This is what is implied by *educating awareness*.

Because students experience the same phenomenon—needing to have an appropriate concept, technique, or strategy come to mind when needed—it is important to accumulate ways of working with students and, by analogy, ways of working with teachers, so that their future practice is enriched by having a wider range of possibilities come to mind. If students become dependent on teachers, or teachers on educators, in order to be reminded to act in ways that could in fact become part of their own repertoire, then learning is not actually taking place.

There is an interesting confirmation of this coming from studies into worked examples. Renkl and colleagues (Atkinson, Derry, Renkl, & Wortham, 2000; Renkl, 1997, 2002) have found that what students really need in a worked example is not simply what to do next at each stage, but how one knows what to do. They offered a range of devices that prove effective in research settings, all to do with prompting the student to reconstruct or to bring to articulation the choices made and what informs them. Something similar applies to learning to teach. When observing an experienced teacher, novice teachers tend not to discern the key moment-by-moment choices being made. Even when they do, they do not have access to how that choice came to the teacher's mind. Simply putting it down to experience is not good enough in a complex society where there is insufficient time and energy available to learn from experience alone.

These insights resonate with our own work. Davis, for example, has organized much of his research program around working with practicing teachers to develop strategies for noticing, deconstructing, and blending the metaphors and metonymies that can be so transparent for expert mathematical knowledge and so opaque for novices (Davis & Renert, 2013). As Lena's example (Account 4) illustrates, such collaborative moments of noticing what might be driving one's mathematical noticing can contribute to a more responsive, in-the-moment pedagogy. Becoming explicitly aware can educate one's awareness so as to increase one's sensitivity to notice more richly in the future, through imagining oneself in typical situations and responding correspondingly.

FOLDING BACK

The four provocations with which we introduced these ideas were for us paradigmatic instances of things that teachers and students say and experience. Alternatively—and drawing on the vocabulary we have invoked to discuss teachers' disciplinary knowledge of mathematics—these provocations are instances that reveal aspects of our own mathematical beings as mathematics knowers, mathematics teachers, and mathematics teacher teachers. They resonated with our experience but also triggered actions.

The provocations, for example, gesture toward the importance of attentiveness to networked connectivity among ideas, among knowers, and among ideas and knowers. In the case of Account 1, B.D. was spurred to work on the connective tissue between mathematical awareness and in-the-moment pedagogy, having noticed the break that can arise when the teacher's attentions are on replication rather than engagement. It is precisely because of the inadequacy of trying to replicate someone else's lesson that we have developed our ways of working with teachers, concentrating on working on mathematics, reflecting on what has been noticed about effective and ineffective actions, and prompting people to imagine themselves acting freshly in the future. Account 4 provides an example of a teacher having an action come to mind, providing appropriate connective tissue.

Accounts 2 and 3 highlight different sites of connectivity. Account 2 provides a justification for working on educating awareness, here on two levels: realizing that "having written something down" is inadequate preparation, being unsupported by preparatory mental imagery and other practices summarized as the discipline of noticing, and realizing that working effectively with teachers requires more than lists of tasks, lists of pedagogic strategies, lists of didactic tactics, and tests of mathematical knowledge. Account 3 was a paradigmatic instance of mathematical knowledge not coming to mind when needed, an example of in-the-moment knowing of mathematical actions for students, as a parallel to similar phenomena for teachers.

As for our own teaching, each of the provocations is an instance of in-the-moment pedagogy. Each event presented itself as a rupture of sorts, a need to pause and interrogate what was being taken for granted that might be channeling perceptions and constraining actions. Each provides reasons for transcending static conceptions of knowledge and justification for working phenomenologically with teachers in order to be effective in prompting them to educate their awareness.

SUMMARY

We delineated three different questions (and sorts of answers) that have oriented the bulk of the research into teachers' disciplinary knowledge of mathematics, and suggested that these sidestep the important question (number 4).

1. What *mathematics* do teachers need to know in order to teach mathematics?
   (Teachers need to know more advanced mathematics than the mathematics they are teaching.)
2. What *specialized mathematics* do teachers need to know in order to teach mathematics?
   (Teachers require a multifaceted specialized mathematics that involves, among other components, pedagogical content knowledge and specialized content knowledge.)
3. What specialized mathematical knowledge *is entailed by the work* of teaching mathematics?
   (Teachers' mathematical knowledge is enacted in their daily work and must be unpacked.)
   Notably, successive questions did not obviate earlier concerns, nor did new answers eclipse the importance of prior insights. Rather, the second iteration of research encompassed and elaborated the first, just as the third included and transcended the previous two. All of them matter. We strive for this same attitude as we move toward our own formulation of a fourth question (and response) that attempts to maintain the inherent complexity of the phenomenon.
4. How might teachers nurture their *mathematical awareness* to enable *in-the-moment pedagogy*?
   (Teachers need to work at developing their through engaging in mathematical thinking for themselves and with other like-minded colleagues.)

For us, then, teachers' disciplinary knowledge of mathematics is more than a demonstrated competence in advanced mathematics, a specialized pedagogical content knowledge, and a set of practiced competencies that arise through years of practice. It is also a way of being *with* mathematics knowledge that enables a teacher to structure learning situations, interpret student actions mindfully, and respond flexibly, in ways that enable learners to extend understandings and expand the range of their interpretive possibilities through access to powerful connections and appropriate practice.

Learning to be responsive through sensitivity to students, rather than depending on reactions derived from habits, requires preparing to be present in the moment. We suggest that any apparent paradox in preparing to be spontaneous is oxymoronic only from a stance of articulated knowing about and an addiction to established procedures as the basis of expertise. From a stance of constantly working to expand and enrich what is noticed, becoming ever more aware of both what one is attending to and in what manner—in short, developing one's own being, the phrase is intensely meaningful. But it points to becoming a teacher as a lifelong adventure, not a short-term activity.

NOTES

1. MKT is perhaps the most prominent and the most generic of the many acronyms used in the current research literature to refer to teachers' disciplinary knowledge of mathematics. Others include MfT and M4T (mathematics-for-teaching), LMT (learning mathematics for teaching), and KQ (knowledge quartet), each of which is used by researchers to signal emphasis on one or another aspect of teacher knowledge.
2. Ball and colleagues (e.g., Ball, Thames, & Phelps, 2008) identified several subcomponents to teachers' disciplinary knowledge of mathematics, distinguishing among, for example, "specialized content knowledge" and "pedagogical content knowledge." We use the term *specialized mathematics* to encompass all components of disciplinary knowledge pertaining to teaching, acknowledging that much of the subtlety of their distinctions is omitted here. Our point is simply that a critical evolution in the field was the recognition that teaching entails a specialized disciplinary knowledge that is distinct from what might be gleaned from more advanced study.

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