Introduction to differentiation

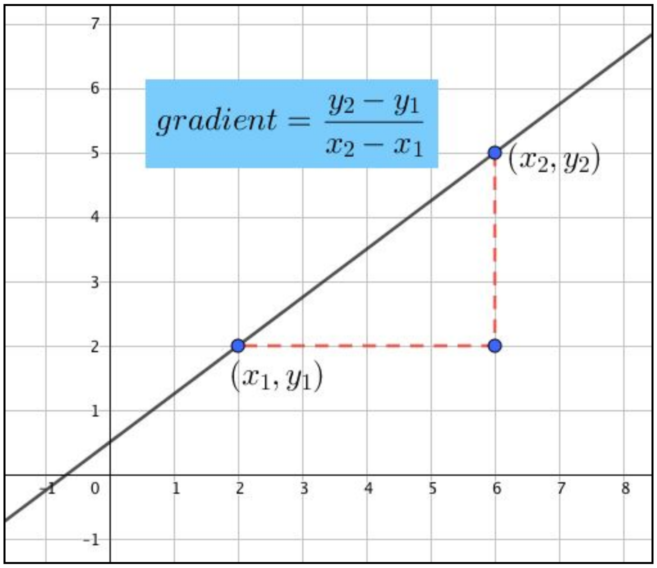
Differentiation is way of finding the gradient of a curve / function, but it is also a way of finding the 'rate' at which something changes. These notes focus on the gradient aspect, and we will come to the idea of rate of change later.

Let's explore what you have learned so far about **gradient**.

You might think of it as the m in the formula $y = mx + c$, or perhaps you think of the formula for calculating the gradient from two points.

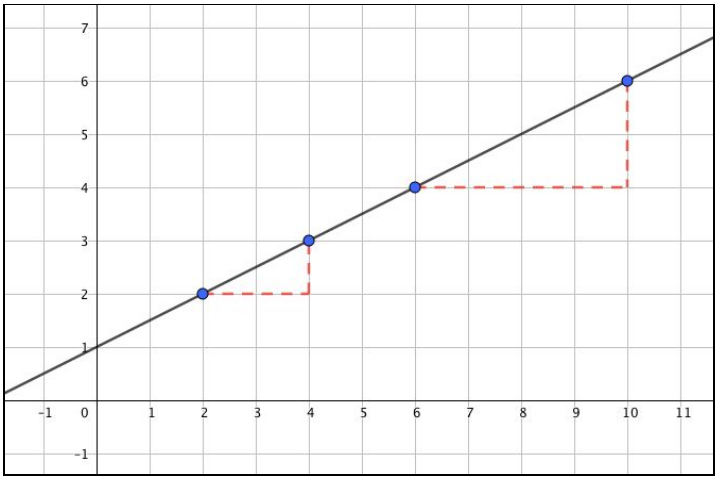
You might remember this as *"change in y over change in x"*.

This is OK for straight lines, but how do you find the gradient for a **curved** line, if such a thing even exists?



The crucial thing about straight lines is that the gradient is the same between any two points on the line, as can be seen from this diagram:

It doesn't matter how or where you calculate the gradient of this line: it will always be ½.



Now, this might seem a bit odd, but although we calculate the gradient **between two points**, it also makes sense to say that the gradient **at any point** is ½.

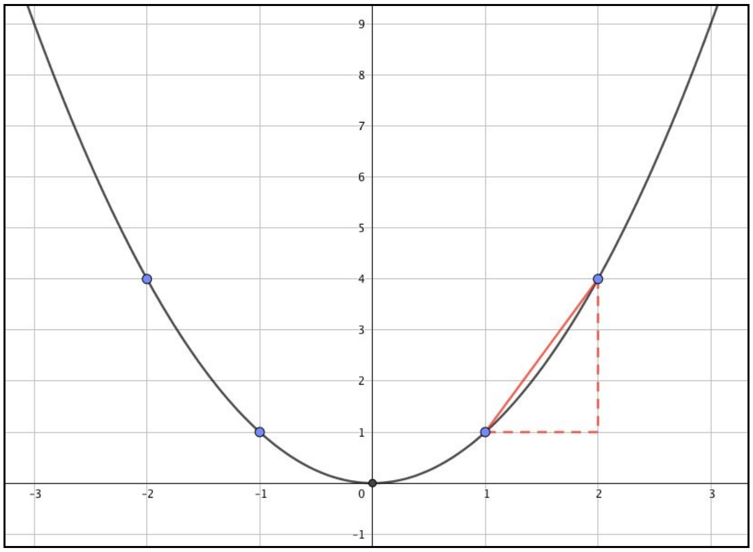
We could say the gradient at $x = 2$ is ½, the gradient at $x = 10$ is ½. It makes sense to say that the gradient is ½ at any point on this line.

But this is **not** true for a **curve**, such as $y = x^2$.

For a curve, the gradient is **different** at different points on the curve. For example, the gradient of $y = x^2$ is **negative** when $x < 0$, but **positive** when $x > 0$. So it matters where we take the gradient.

Secondly, the curve is not straight. So does it still make sense to use the formula $\frac{y_2 - y_1}{x_2 - x_1}$?

The short answer is **no**. Look at the picture on the right, in which I have attempted to use the formula to find the gradient between the points $(1, 1)$ and $(2, 4)$.



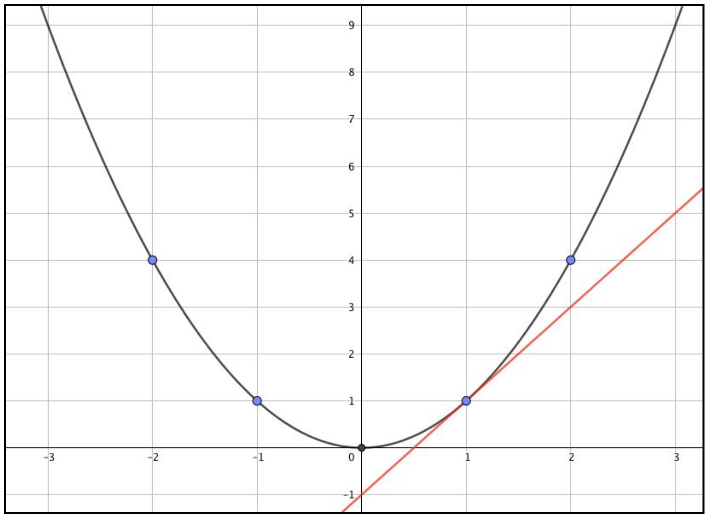
What does this tell us about the gradient of $y = x^2$ at $x = 1$? Or $x = 2$?

The gradient can't be 3 at both points, because the curve is steeper at $x = 2$ than $x = 1$.

So what do we *mean* by finding the gradient at each point, and how *do* we do it?

Firstly, we need the bring back the idea of a **tangent**. We have encountered tangents in our work with circles. A tangent is a straight line that just touches a curve.

If I draw a tangent to the curve $y = x^2$ at $x = 1$, it might be obvious to you that the tangent has the same gradient as the curve at the point $x = 1$.



If we look at this tangent, it has gradient 2, and we say that the curve $y = x^2$ has gradient 2 at the point $x = 1$.

So one way of finding the gradient of a curve at any point could be to draw a tangent, but this is not a very accurate way.

There is a better way, but don't forget this tangent idea as we will come back to it later.

Fortunately, two mathematicians called Newton and Leibniz came up with the idea of **differentiation** for **calculating** the gradient of a curve at any point.

Before we learn about **why** differentiation works, we are going to find out **how** to do it, check it makes sense, and look at the theory afterwards.

Differentiation is quite simple, although you might get asked to differentiate complex looking functions, and solve some tricky problems using it. Basically, there are only two types of function you will be asked to differentiate:

Functions involving powers of x	Trig functions
$x^n \xrightarrow{\text{differentiate}} n.x^{n-1}$	$sin(x) \xrightarrow{\text{differentiate}} cos(x)$ $cos(x) \xrightarrow{\text{differentiate}} -sin(x)$

So, for example, x^2 differentiates to $2x$. In words, you might say: *"If you differentiate x^2, you get $2x$."* Or you might say: *"$2x$ is the **derivative** of x^2."*

What has all this got to do with finding the gradients of curves? Well, the derivative of a function is also (!) called the **gradient function**. So, the gradient function of x^2 is $2x$.

Now, it turns out that **if we substitute the value of x for which we want the gradient into the gradient function, we get the gradient at that point!**

So, in our example, if we wanted the gradient of x^2 at $x = 1$, we just substitute $x = 1$ into the gradient function $2x$, which gives a gradient of $2 \times 1 = 2$. This is the same gradient as our tangent above!

We could find the gradient of $y = x^2$ for **any** value of x, just by substituting the value for x into the gradient function $2x$. For example, the gradient of $y = x^2$ at $x = 2$ is $2 \times 2 = 4$, or at $x = -3$ is $2 \times -3 = -6$. You can check these sort of make sense by looking at the graph above.

So, to summarise: **We can find the gradient of any curve at any point by differentiating. This gives us the gradient function. We can substitute any value for x into this gradient function to find the gradient at this point.**

A last word on notation. If we are talking about curves, such as $y = x^2$, then we use the notation $\frac{dy}{dx}$ for the derivative. So if $y = x^2$, the derivative (or gradient function) is $\frac{dy}{dx} = 2x$.

But if we are talking about functions, such as $f(x) = x^2$, then we use the notation $f'(x)$ for the derivative. So if $f(x) = x^2$, the derivative, or gradient function, is $f'(x) = 2x$. This notation has the advantage of saying that if we want the gradient at (say) $x = 1$, then we can state this as $f'(1)$, which we saw above is 2.

Now we need to do lots of practice differentiating and finding gradients of curves, then solving problems involving these, before talking about rates of change, or finding out **why** it all works. In fact, you will not be examined on the why, but it might be useful to know where all of this comes from.