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Towards proof

Dave Hewitt suggests that proof is important in school mathematics and has some suggestions about giving it more weight in the work of the classroom.

I consider proof to be important. There, a nice easy statement to make. However, why do I think this? Why do I think that proof is so important that I would encourage proof to be on the curriculum for every classroom no matter what age the students are and no matter what attainment the students have achieved. Proof is concerned with properties. Any proof of why something is so makes use of accepted properties in order to establish a new property. Sometimes these properties are algebraic, as in the following example:

Proof that the addition of any three consecutive whole numbers is a multiple of three:

Let x be the lowest of these three numbers, then the other two numbers will be $(x+1)$ and $(x+2)$.

Adding these three together gives:

$$\begin{aligned} x+(x+1)+(x+2) &= 3x+3 \\ &= 3(x+1) \end{aligned}$$

This is three times a whole number and so is a multiple of three.

This 'proof' makes use of several properties which are assumed to be unproblematic. For example, the step which turns $x+(x+1)+(x+2)$ into $3x+3$, makes use of properties of addition such as associativity and communtativity. Any proof always makes use of some properties which are known to the writer and assumed to be understood and accepted by the reader of the proof. These properties have been used in order to produce a new property, which is that the sum of three consecutive whole numbers is a multiple of three. In fact, more can be said through an awareness that the final expression $3(x+1)$ represents three times the middle of the three original numbers. This realisation can only come through a way of seeing the final expression. Attention might be placed with the '3' in $3(x+1)$, because it is this which indicates the result is a multiple of three. However, if attention is shifted to the $(x+1)$ in the expression, then there can come a new awareness of which multiple of three the result is. Where attention is placed and of what one is aware are important within any proof. Proof is not only about properties, but is also about awareness of properties. If I am trying to follow a proof then I can only know that it proves something if I am aware of the properties which are used in the proof. Examining Andrew Wiles' recent proof of Fermat's Last Theorem would not help me to know that this theorem is true. I expect I would not be aware of most, if not all, of the properties used within the proof. Since I am in this position, the only thing left for me is to have trust that what he has written is correct, and that those who have checked the proof are correct in saying it is right. In terms of my own knowledge of those things which I know to be true and can justify,

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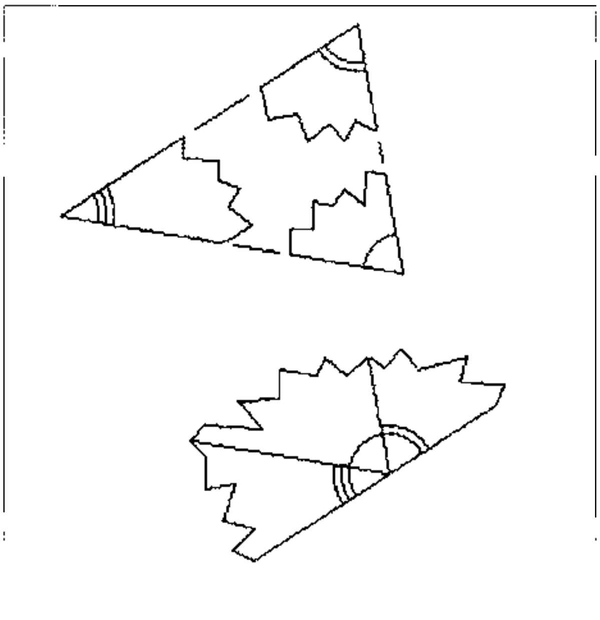
the fact that someone has finally proved Fermat's Last Theorem has made little difference. The closest I have got to knowing anything around this theorem is that I have not been able to find any whole numbers for a, b and c such that:

$a^3 + b^3 = c^3$ or
$a^4 + b^4 = c^4$ or
$a^5 + b^5 = c^5$ etc...

I have tried several numbers and never succeeded. The only awareness I have got is that I have not been able to find a solution. Furthermore, I know that every student I have given this problem to has not found a solution. I am aware that a solution is difficult to find, but I am not aware that a solution has to be impossible. The awareness I have gained may help me believe that a solution is impossible, but I do not know that through my own awareness. It only helps me to accept someone else's word. It does not change that fact that I still have to accept someone's word.

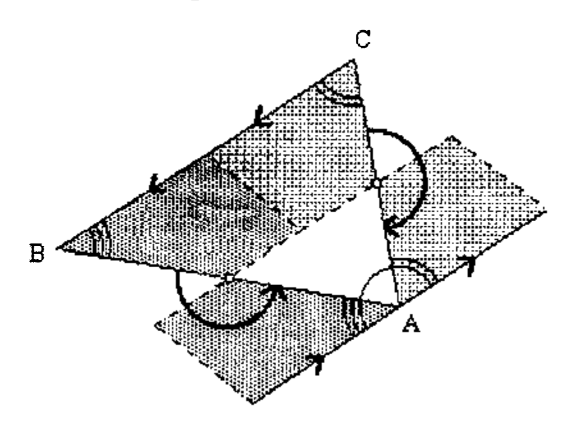
I have seen an activity, frequently used in mathematics classrooms, of cutting out different triangles from paper, then cutting their corners off and putting the corners together.



Doing such an activity can lead to an awareness that the angles seem to make a straight line, and that this happens for several different triangles tried. But often, that is all that is achieved by this activity. There is only an awareness that it is difficult to find a triangle which does not have its angles looking as if they make a straight line. Perhaps no-one in the class has found an exception. This can lead to a sense that *possibly* all triangles have their angles making a straight line. So a student might be more prepared to believe the teacher saying that this happens with every triangle. However, it is still on the level of trusting what someone else says is true, rather than a student knowing that it *must* be true through the awareness he or she has. There is a qualitative difference between someone being able to know without the need to be informed by some external source (such as a teacher, book, CD ROM, etc), and someone only knowing because what is read, or what the teacher says is trusted. Proof is only concerned with the former.

Consider another situation where a class is being asked to find a perfect number, a number where the sum of all its factors (excluding the number itself) adds up to that number. Students might spend a whole lesson trying different numbers and not find a single one which is perfect. What is the difference for a student between this experience and the one of trying to find a triangle whose angles don't make a straight line? There is little difference between a problem not having a solution at all, and there being so few solutions that they are very hard to find. Trying lots of particular cases will not lead to knowing for sure whether something has to be so.

The activity of cutting out the corners can offer a way of seeing that they *must* add up to 180 degrees. However, it only does this if properties are considered. For example, if corners B and C are each rotated as indicated by the diagram below, then there are several properties which can be known through awareness. If the centres of rotation are the midpoints of the sides AB and AC, both corners B and C will be rotated exactly onto the third corner, A, if the angle of rotation is 180 degrees. And if the rotation is through 180 degrees, the two halves of the line BC will each be rotated by 180 degrees and end up pointing in the same direction. So, they do make a straight line through A. This means that the three angles situated round A must form a straight line and so add up to 180 degrees.



I haven't expressed this argument particularly formally but the point I wish to make is that all this can be known through awareness. It can be known, for certain, that the angles add up to 180 degrees exactly, but only through paying attention to properties. If properties are not considered, and students just look at the three angles together and say yes, that looks like a straight line, then those

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students will be a long way from proving that it is the case. It seems to me that an important role a teacher has to play is to help students attend to properties, which means that a teacher needs to focus on properties, and any activity needs to focus on properties.

Consider another situation. Below is a table of results obtained *from an investigation:*

Number of pegs each side (p)	Number of moves (m)
1	3
2	8
3	15
4	24

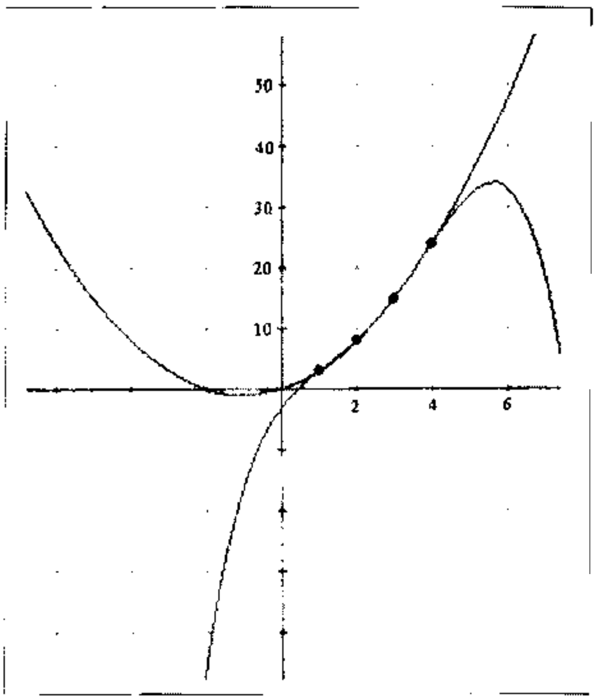
Before reading on, consider how you would prove that the formula which connects the number of pegs on each side and the number of moves is:

$$m = p^2 + 2p$$

and not

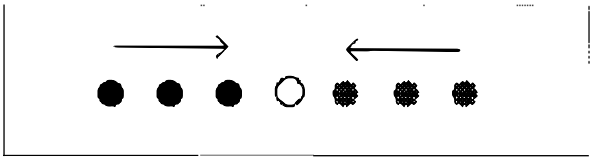
$$m = -\left(\frac{p^4 - 10p^3 + 27p^2 - 66p + 24}{8}\right)$$

Both of these formulae give the same values as those in the table. In fact there are an infinite number of non-equivalent formulae which will produce the values in this (and any other) table of results. After all, there are an infinite number of curves of functions which go through four points on a graph.



To prove that a formula is correct requires more than showing that the formula fits in with the results from an investigation. With the only information being the numbers, there is no way to know whether one formula is right and a different one is not. They all could be right as far as the data are concerned. So, proving that a formula is correct for a particular situation cannot be achieved through looking purely at the data generated from a few special cases. Proof can only lie in looking at the situation itself. Attention needs to be turned away from the numbers and onto the situation where those numbers came from. Yet so many 'investigations' take the form of asking students to collect results from individual cases, put them into a table, and look to see whether they can find a rule. There is an implied *find a rule... from these numbers* rather than *find a rule... from the situation where the numbers came from*. There are lots of formulae (not equivalent to each other) which will fit any set of numbers, but only one formula (or its equivalents) which represent a particular situation and a particular interpretation of that situation. How are students expected to be able to prove something if their attention is taken away from where a proof may be found and encouraged to attend to where it is impossible to know why a formula is correct?

If I now say that the above results come from the well used situation of 'frogs' where there are a certain number of frogs, or pegs in this case, each side of a gap, the idea being for the pegs to change positions – black pegs where the shaded pegs are, and vice versa. The moves are that a peg can move one space into an empty space, or jump over one of the different coloured pegs into an empty space. What are the smallest number of moves to get the pegs to swop places?



The diagram shows the situation where there are three pegs each side. The investigation is to consider what is the minimum number of moves for one, two, three, etc pegs on each side. Now that you are aware of the situation where the numbers came from, you are in a position to justify why a particular formula is correct, or indeed to find a formula in the first place. For example, one way I worked on this problem was to say that each peg had to move past the empty space and the number of spaces occupied by the pegs on the other side, which is $p + 1$ for each peg. There are $2p$ pegs altogether (p on each side), so there are $2p(p + 1)$ moves required. However, each of the black pegs have to jump over each of the shaded pegs (or vice versa – there is one jump for each pairing of black peg with shaded peg). There are p black pegs and p shaded pegs, and so there will be $p \times p = p^2$ jumps. Each time a jump is made, a peg manages to move two spaces rather than one space, so one move is saved. So the total number of moves is $2p(p+1) - p^2$. This gives me a formula which is equivalent to $m = p^2 + 2p$ and has come through an awareness of properties of the situation (rather than properties of numbers from the situation), which are things which I know have to

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be true, and I can justify. There are many other ways in which this situation can be viewed which would lead to other formula, but they would all be equivalent to $m = p^2 + 2p$, none of them would result in the quartic expression I offered above. That quartic fits the numbers in the table but does not fit with the situation itself.

Some activities in the classroom take students' attention away from properties of a situation and away from any chance of being in a position where they can prove something. The use of a spreadsheet can do this, even though it can provide an efficient way to answer a problem. For example, consider tackling the following problem with a spreadsheet: *a rectangle has an area of 20 sq. cm. and a perimeter of 20 cm. What are the dimensions of the rectangle?* Tables can be produced such as those below, where a solution is found which the length between 2.76393 and 2.76394 by 'homing in' further and further.

Length	Width	Perimeter	Area
1	9	20	9
2	8	20	16
3	7	20	21
4	6	20	24
5	5	20	25
6	4	20	24
7	3	20	21
8	2	20	16
9	1	20	9

Length	Width	Perimeter	Area
2.1	7.9	20	16.59
2.2	7.8	20	17.16
2.3	7.7	20	17.71
2.4	7.6	20	18.24
2.5	7.5	20	18.75
2.6	7.4	20	19.24
2.7	7.3	20	19.71
2.8	7.2	20	20.16
2.9	7.1	20	20.59

Length	Width	Perimeter	Area
2.76391	7.23609	20	19.9999
2.76392	7.23608	20	19.99995
2.76393	7.23607	20	19.99999
2.76394	7.23606	20	20.00004
2.76395	7.23605	20	20.00008
2.76396	7.23604	20	20.00013
2.76397	7.23603	20	20.00017
2.76398	7.23602	20	20.00021
2.76399	7.23601	20	20.00026

This produces an approximate solution to the problem and involves work thinking about what formulae are needed in each column (which will depend upon the approach taken). This does pay attention to some properties of a rectangle. However, once the formulae are put into a spreadsheet, it is likely that attention will be taken by the numbers and it becomes an activity about numbers and properties of numbers. This may provide useful practice for number work, and develop important technology skills; however, it must also be realised that it is taking attention away from properties inherent within the original problem. Once someone has found a numerical approximation to the solution, what insights has this brought to the original situation? How close is someone to being able to prove that their answer is correct? How would someone know for sure whether there is another solution, or whether the solution obtained is unique? I suggest that although the use of a spreadsheet can offer opportunities for such things, it will require someone resisting the seduction of numbers. Looking at the first table, for example, and saying that it looks as if there are two solutions: one with the length being between 2 and 3 and another between 7 and 8, is not a proof. How does someone know for sure that there are not other solutions when all that can be said is that it doesn't look as if there are any others? Proof requires argument and reasoning to justify statements. Furthermore, the justifications must be based on the situation being investigated. For example, an attempt at a justification based on the numbers in a table, such as *these were the only places where the numbers changed from being below 20 to above 20 for the area*, is failing to consider that the table does not present the whole situation. Maybe another table will show other situations where the numbers change from below to above 20. The only time when numerical data are useful for proving something is when there are a finite number of cases to consider. This can lead to proof by exhaustion, proof by considering every possibility. However, there still needs to be justification that all possibilities have been considered, and this justification will be based on the properties of the original situation. Even when someone does prove something by proving an equivalent problem, as indeed Andrew Wiles did with Fermat's Last Theorem, what has to be included in a proof are the connections between the original problem and its equivalent, and a demonstration that the two problems are exactly equivalent.

So, a proof is always going to be based on the properties of a situation, those things which someone knows must be true. Since we are all individuals and have different things which we know to be true, it follows that the properties of which you are aware may well be different to the properties of which I am

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aware. If I try to prove something, then not only must I base it on properties of the situation I am trying to prove, but I can only base it on properties of which I am aware. If my teacher presents to me a 'proof' which is based on properties I do not know, then it will not have done the job of proving something as far as I am concerned. In the end, I will have to trust (or not) that what my teacher tells me is true, rather than knowing it is true through using the awareness I already have. This puts the subject matter of the proof as something which I will only know through memorising rather than understanding.

If students are to know about proof, then they will have to come to know about it through using their own awareness rather than trusting what someone else says is true. It is for this reason, that I have approached proof with students using something with which they are already very familiar. On the two-year PGCE course I teach, with students who already have an A level in mathematics, I use the 'fact' that there are 180 degrees in a triangle. This is a result with which they are very familiar and is in an area of mathematics which they know quite a lot about. This means that they already have considerable awareness which can be employed in trying to prove this result which, up until now, they have just accepted as a given and never questioned. It will have to be up to you to decide an equivalent fact which could be used with your students. In trying to prove this familiar result, I act as someone who does not accept everything that is said but tries to be helpful in pointing out an assumption made or a claim which is not supported by a justification. After a while, students see the role I am playing and begin to notice such things in what others say and gradually take over from me. As I withdraw from that role, I begin to offer some meta-level comments, bringing out such issues as: that there are always assumptions which an argument is based on; that sometimes the original problem is transferred to an equivalent problem (solve this and you've solved that); that finding something is true for particular cases does not mean it is true for all cases; that there is a difference in believing that something is true and knowing that it is true; that because arguments sound convincing it does not mean that they are true (many 'proofs' have been accepted and then found to have a mistake in them); that there are different forms of proofs (by exhaustion, by contradiction, etc.). Over a period of time, many students shift their attention onto these meta-issues, and it is then that I feel the students are becoming aware of proof. Being aware of proof is more than being able to prove something, it is being aware of the nature and form of proof rather than the details of a particular proof. The same is true in other areas of mathematics. To be aware of multiplication requires more than being about to carry out one or two particular multiplications. Becoming aware of proof, as opposed to proving something, leads to an increased ability to prove other things in the future and know when something has not yet been proved. This is important, because it is often the case that in the creating of an argument one attends to the argument rather than any loop-holes in the argument. Attending to proof can force a student to examine what they do know and what they do not know about something. This forces attention onto properties, and it is not unusual for someone to learn a number of new things on the way to a proof. I believe it is the attention to properties which helps develop someone's learning in mathematics, because that is what mathematics is about.

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