Grace

Introduction

Aware that the total universe is shrouded in mystery, the mystic learns to respect the whole… The mystic attributes his contact with the whole to a special favor unrelated to his capacity to reach it, one which is beyond him in terms of all he did to 'deserve' it. That is why he speaks of 'grace'.

Gattegno, The Science of Education

When we are truly stuck on a problem, *insight* is required. Where does insight come from? Are there actions we can undertake that will make us more likely to receive insight?

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Consider the following two problems:

Problem 26. Suppose 5 points are given in the plane, not all on a line, and no 4 on a circle. Prove that there exists a circle through three of them such that one of the remaining 2 points is inside the circle while the other is outside the circle.

Problem 27. Let ABC be an arbitrary triangle, and P any point inside. Let d_1, d_2, and d_3 denote the perpendicular distance from P to side BC, CA, and AB respectively. Let h_1, h_2, and h_3 denote respectively the length of the altitude from A, B, C to the opposite side of the triangle. Prove that

$$\frac{d_1}{h_1} + \frac{d_2}{h_2} + \frac{d_3}{h_3} = 1.$$

It would be beneficial to try them both before reading on, as I will be describing what happened when I attempted to solve them.

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Did you get stuck on either or both?

I managed to solve the first one, but got stuck on the second one. In essence, this was because insight appeared for one and not the other.

Was this *purely* a matter of luck, or was this due, at least in part, to my actions? What actions might allow or obstruct the possibility of grace?

In this article, I explore actions we may take to in order to free ourselves to receive insight.

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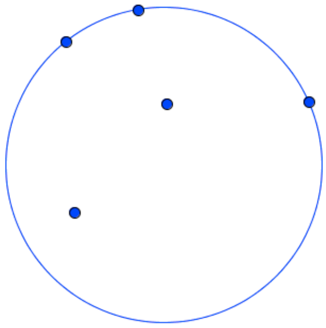
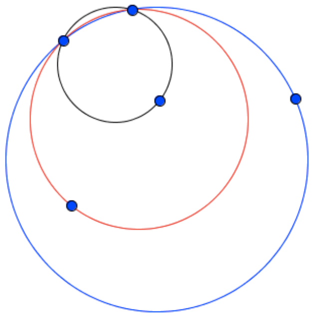
Problem 26

Having tried a few examples and drawn various circles through them, it became clear that some circles are larger than others, and that there must be a biggest circle, containing both of the other points.





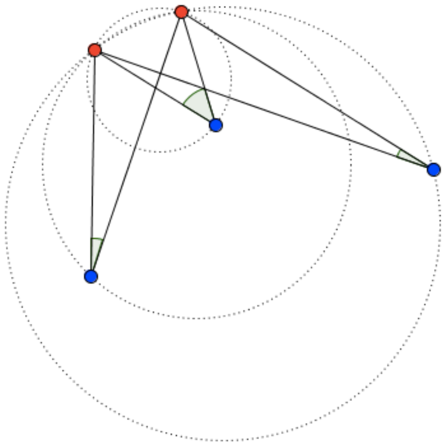
Choosing two of the points on this biggest circle, we can then form two smaller circles through each of the other two points.

This suggests that there is a 'middle' circle, with one point either side of its boundary (shown in red).

Is this a satisfactory proof? I was not convinced, and began contemplating the diagram.

This led to the idea of the two common points forming an 'arc', which in turn led to this a sketch.

In an instant I 'knew' that this provided a rigorous proof.



We can always choose two points (shown in red) and construct angles as shown with the other three. These angles must be different, and so there is a middle one, and so we have a proof.

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This final insight, using a theorem from GCSE geometry, appeared in what felt like a moment of grace.

Whilst it is true that there must be something known (the circle theorem) for the insight to occur, and that I had prepared well by sketching various examples and maintaining a contemplative state of mind, the final insight - the connection between the theorem and present experience - appeared in a flash, by some mechanism which cannot be induced.

It would seem that we can prepare in various ways for insight to appear, but we cannot make it happen. Is this all we can say?

Problem 27

I worked on problem 27 for around an hour, and did not reach a solution. I tried various examples, diagrams, considering various aspects of mathematics. I was generally calm, but no insight appeared.

At one point, I wrote the following: *STUCK! Doubt: Do I 'have the mathematics' to solve this?*

We may try various approaches but insight does not arrive. With this comes doubt, which brings with it negative emotions. There may be frustration, and the urge to give up. We start to wonder whether the problem is just beyond our limitations.

How can we know whether we 'have the mathematics' to solve this problem? The fact is that we can't. It is in these moments that we learn most.

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I remind myself that I am solving these problems in order to learn *about the getting stuck*. I am aware, however, that for many - especially students - it is primarily *about solving the problem*.

It is easy for me to take this attitude, as I am choosing to solve the problem to learn something about myself, whereas students may not have this choice, particularly those studying for exams.

But I would argue that it is as important to observe the *dynamics* as well as the *content*. If we only focus on the content we lose an opportunity to learn something important that may be useful in any situation, including - and perhaps especially - in exams.

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When doubt arises, a useful action might be to step away from the problem. This is difficult. We have become involved, we have invested time. But carrying on often ends in repeatedly going over the same old ground.

I find it useful to call on meditation techniques here. With practice, we can almost instantaneously detach from the problem at hand by focusing on our breathing. This brings a return to the present moment, and calm. When we have become more clear of mind, we may return to the problem.

Of course, this does not guarantee a solution, but I have found it improves the chances of receiving insight.

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With this problem, there was an elementary aspect of geometry that I had not considered.

I certainly did 'have the mathematics' to solve it, and various problem solving techniques to draw on, but in this case I had not managed to make the connection between something known (a fact that I 'know' very well), and the problem in front of me.

We can put this down to bad luck - such is the nature of grace. But perhaps there is something more to learn here.

I believe that I was hindered here by the suspicion that I did not 'have the mathematics'; this may have obstructed the flow of ideas, and led me to suspect the problem was more complex than it turned out to be. It will be beneficial in the future to assume that I do 'have the mathematics', and that it might only be a matter of finding the connections between what I know and the problem that is in front of me.

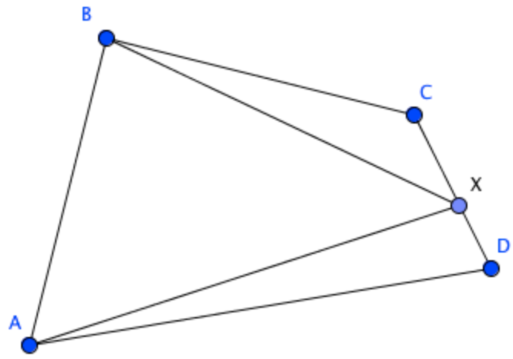
A final problem

Let ABCD be any quadrilateral, and X any point on the side CD as shown.

Draw a line parallel to BX through C and a line parallel to AX through D.

These two lines intersect at P.

Prove that the area of the quadrilateral is equal to the area of triangle ABP.



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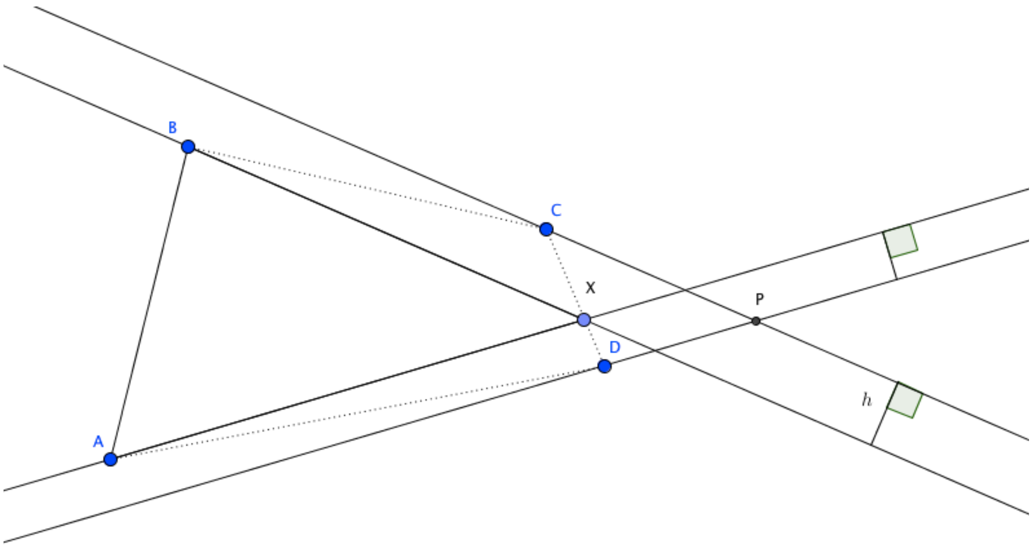
We can approach this problem in the knowledge that we may or may not be able to solve it, and that even if we don't solve it, we can learn something important about solving problems, and about ourselves, beyond the content.

Do we 'have the mathematics' to solve this problem? We can't know either way, so *we may as well assume that we have*. This brings a confidence.

There are things that make this problem difficult. ABCD is *any* quadrilateral. We do not know how to find the area of a general quadrilateral. This suggests we might (1) choose a quadrilateral we know something about and see if that provides any insight, but perhaps more importantly (2) that we are probably not going to be expected to find the area of ABCD directly (i.e. using formulae), which gives its own insight.

The essence of the problem is in the construction of two pairs of parallel lines. Does drawing a diagram which *stresses* the parallel lines help?



We want to show that the area of ABCD is the same as the area of ABP. The addition of this triangle onto this diagram will certainly give further insight.

Can you solve it?

Conclusion

There is no guarantee we will be able to solve a problem, even if we 'have the mathematics'. We can do all we can to prepare for insight; sometimes it appears and sometimes it does not.

However, we can always guarantee that we will learn something about solving problems and ourselves. If we approach problems with this frame of mind, attending jointly to the dynamics as well as the content, we will certainly learn something from the experience.

It is helpful to assume that we 'have the mathematics' to solve the problem, as assuming otherwise may lead us to adopt a more complex approach than is required.

Once we have prepared as well as we can for insight, it is helpful to enter what I call the 'contemplative state', to view the problem as a whole from as many different perspectives as possible - and then wait, with equanimity, for a moment of grace.