Removing the Shackles of Euclid

1 Classification

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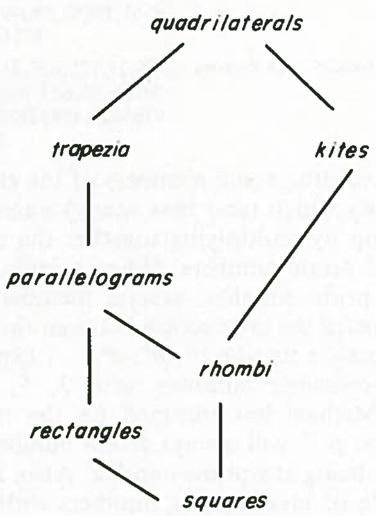
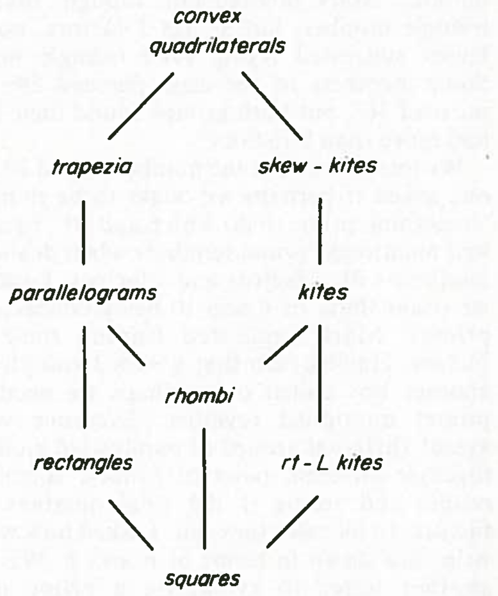
How can you classify quadrilaterals?

Eric Love asked the question of his seminar group at the ATM course in 1976, and a brief outline of the discussion was published in the report in *MT* No. 75. But as far as I know, no-one has taken up the final idea.

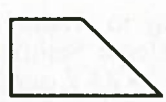
At the time we were initially bogged down by the familiar names, which involve a classification based on Euclidean ideas of parallel sides, equal sides and right angles. Yet this is such an untidy classification, as Philip Lewis showed [1] when he sought to complete the symmetry of this otherwise apparently nice-looking hierarchical diagram.



Quadrilaterals become trapezia if *one* pair of sides become *parallel,* but quadrilaterals become kites if *two* pairs of sides become *equal.* Philip formed what he called 'skew-kites' by making just *one* pair of sides *equal,* and then completed the symmetry of the diagram by inventing 'right-angled kites',



the right-angle being enclosed by a pair of equal sides. However, he was still not satisfied, since there was an intersection of skew-kites and trapezia.



Furthermore, the pleasantly symmetrical diagram, he conjectured, was part of a yet more complex one, "the aim being to trace all possible conversion paths from quadrilaterals to squares".

Well, to do that one must decide what conversions are permitted. Given his three, we should add the right-angled trapezium above, which is also a right-angled skew-kite and which can be converted to a square by creating another right angle *or* another pair of equal sides *or* another pair of parallel sides! The diagram becomes complicated already. Would you care to have a go with colour-coded arrows?

There were theorems anyway, and since Philip also used some complicated Venn diagrams to throw them up I can paraphrase them thus (we may as well use all that stuff on sets we did in Book 1, Chap. 2!):

$$\{\text{rhombi}\} \cap \{\text{rectangles}\} = \{\text{squares}\};$$
$$\{\text{kites}\} \cap \{\text{rectangles}\} = \{\text{squares}\};$$
$$\{\text{kites}\} \cap \{\text{parallelograms}\} = \{\text{rhombi}\};$$
$$\{\text{kites}\} \cap \{\text{trapezia}\} = \{\text{rhombi}\}.$$

Let me digress for a moment. Or rather, since I have really begun with a digression, let me now approach my main theme.

Most of us have been conditioned to think about geometry in a particular, Euclidean way, as illustrated by Eric Love's seminar group. So, when asked to classify quadrilaterals we only think of the ones we know, the ones we can name. True, the question makes us think about some of the relationships, but we still feel uneasy about creating new types, and finding appropriate names for them. Familiarity also breeds a certain amount of contempt, as exemplified by attempts in the early days of Venn diagrams to draw one for triangles; most teachers could *name* equilateral, isosceles, scalene, acute-, obtuse- and right-angled triangles, without at first realising there were two distinct classifications – one based on equality of sides and the other on the size of the largest angle – or that two of the cross-categories were empty. (Incidentally, how does one *prove* that the angles of an equilateral triangle are each 60°? That is, what assumptions does one begin with?)

There is thus a theorem, that equilateral triangles can only be acute-angled. It is not a theorem that fits into a Euclidean development, although one can find a proof based, say, on a theorem about isosceles triangles. But it *is* a theorem thrown up by an attempt to draw a suitable Venn diagram for triangles. In the same way Philip Lewis produced some theorems about quadrilaterals, fairly simple as yet, but there is more to come.

However, let me digress again, although I am not sure whether I am now continuing with theme or variation!

Most of the attempts to displace Euclid in school mathematics in the sixties (and I mean Euclid, not euclidean space) did little more, in a way, than provide alternative descriptions. Whether it was vector geometry or motion geometry, the early mistake was to bully the new theory into proving the same old Euclidean theorems, with sometimes most awkward consequences. Thus M.M.E. set out to prove the midpoint theorem using algebraic properties of vectors which assumed the midpoint theorem; and someone from SMP was once delighted at proving the circle theorems using motion geometry, even though it was much more difficult that way, but after a clumsy attempt in *Book T4* the idea was dropped.

If one is to construct a new development of geometry, even a new description, then there must be in general a new set of theorems which arise appropriately, and it is inconvenient, sometimes erroneous, to try to prove the familiar set of theorems which is only appropriate to a Euclidean development. This is a point worth making, but not worth labouring, since on the whole I think it has been taken by compilers of 'modern' geometry syllabuses.

However, what still dismays me is the 'traditional' geometry syllabuses, not because they stick to a synthetic geometry which is largely Euclidean, but because they are subjected to such arbitrary restrictions. I cannot, for instance, understand why the circle theorems appear in C.S.E. syllabuses; they are difficult for average pupils to understand, they are dull, and they are fairly useless. But, more basically, any such syllabus usually begins with a restriction of polygons to triangles and quadrilaterals, and usually states "properties of" followed by a prescriptive list of the familiar names.

Well, there is not a lot to be done about the triangles, except make the cross-classification clear, although it would be better to explore the possibilities and come to a realisation that the *only* two variables are relations between sides and size of angle, and that is in a way slightly sophisticated.

But when we come to quadrilaterals, as Philip Lewis has shown, the traditional classification is a mess. Let me extend his suggestions a little further.

One simple classification is merely by the number of pairs of parallel sides, 0, 1, or 2, which gives us quadrilaterals, trapezia and parallelograms. Now cross-classify this with the number of pairs of equal sides, also 0, 1 or 2.

PAIRS OF PARALLELS

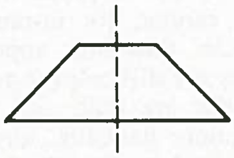
PAIRS OF EQUALS	0	1	2
0			
1			
2			

Would you like to enter appropriate names in the cells before you read on?

Four of the cells will be empty, which means we have four theorems. Well, no, it is only one theorem, which really corresponds to cell (2,2), and it is a fairly familiar one.

Cell (1,1) is the interesting one, because we have a choice about whether the pair of sides which are parallel is also the pair of sides which are equal, or not. If the former, then we have a parallelogram which goes in (2,2), and we need another familiar theorem. If the latter, however, then we have not only a *theorem* but also a *problem* about a *construction,* and also a problem about a name if you have not met it before.

Let me digress again. One idea I have suggested to children is to classify quadrilaterals according to symmetry. They usually begin with axes of symmetry, realising sooner or later that these can join either opposite corners or midpoints of opposite sides (it helps to look at the square first). Apart from the square, this produces rhombus and kite, and rectangle and – lo and behold! – the isosceles trapezium,



which of course was the one to go in cell (1,1) above, but can you supply an appropriate theorem and proof? It is easy to forget the parallelogram this way, because it is the only quadrilateral with rotational symmetry only. But it is difficult (for children) to see that a quadrilateral cannot have *four* rotations of 90° without also being a square and having reflective symmetry as well. (Related problem: draw a shape which has 4-fold rotational symmetry without having reflective symmetry.)

Older/brighter children tackled the same classification problem by looking at the table of products for the symmetry operations of the square and finding all the sub-groups! There are some duplicates to be sorted out, but they obtained all possibilities this way, and also learnt some interesting things about subgroups.

Let us return again to sorting out the Euclidean classification. Let us cross-classify the number of parallel sides with the number of right angles. We can deduce that 3 right angles is impossible, and we know what happens with 4.

PAIRS OF PARALLELS

RIGHT ANGLES	0	1	2
0			
1			
2			

Can you fill in the cells?

Well, perhaps we made the wrong assumptions about ignoring 4 right angles, because if a parallelogram has one right angle then it automatically has four! (Proof?) And we also discover that if a trapezium has one right angle then it automatically has two. The right-angled quadrilateral is easy enough, and not very interesting. The two right angles (bottom left) have to be opposite each other or we produce a trapezium; is that a new theorem or the converse of the other one? And, if you are interested, the quadrilateral with two opposite right angles is also cyclic, but that does not concern us now. (However, a question another time is, which of all these quadrilaterals are, or can be cyclic?)

There are again several theorems, and most of them can in fact be proved using first-year Euclidean geometry; after all, we are using Euclidean properties. But we are not using a Euclidean cross-classification – so the theorems are generally new ones.

What is important anyway is that it is the *children* who do the classification, rather than merely learn a classification that we have made for them. Thus they learn something about the process of classifying and make decisions about choices, facing the consequences of those choices, and making alterations where necessary. In doing this they also learn something about the properties, but in a much more meaningful way.

There remain two things for the current problem. The first is to cross-classify pairs of equal sides with numbers of right angles. Now it will have occurred to you already that our previous consideration of equal sides was inadequate, since we assumed that they were opposite; and this way we are not going to obtain any sort of kite. The kite in the diagram above, for instance, has two sides parallel, and two sides equal but adjacent. So, can you design an appropriate table and fill it in? (All the 'kites' now become particularly interesting.) Should you also redesign the first table?

Finally, find a way of combining all three properties, and then fit Philip Lewis' diagram into it, and thus complete it for him.

It is not easy to break out from the conventions of either Euclid or symmetry in order to classify the quadrilaterals in a different way, but perhaps it was thoughts about equality of sides, and suspending the obsession with *pairs* of sides, that made me suggest in Eric Love's seminar group that we classified merely according to lengths of sides.

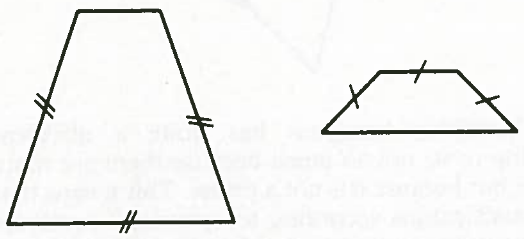
We came up with seven possibilities:

abcd, aabc, abac, aabb, abab, aaab, aaaa.

It needed a realisation that in some cases the order mattered. This is a nice activity in itself for children, using perhaps geostrips [2] or meccano, and is a slightly more sophisticated version of one of an earlier set of problems [3]: given a certain number of lengths available, how many distinct quadrilaterals can be made?

Three of the seven are obviously kites, parallelograms and rhombuses, and we know the properties of those already. But what properties can we now look for generally? In the group we looked first at possible numbers of right angles. Parallelograms and rhombuses could be rectangles and squares with four. Kites could have either one or two. Is it true that, in general, each of the others can only have one right angle? And what do we mean by *in general?* One needs the geostrips in order to get a feeling for the dynamics involved. But here is a rich field for exploration based on something tactile and mobile, with possibilities for theorems and proofs.

We looked at parallel sides. In two of the seven there are *always two* pairs, and in one there are *always none*. In the rest, the theorem is that it is always *possible* to make one pair parallel. In the cases *abac* and *aaab* it is fairly obvious which;



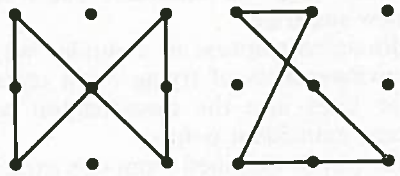
but in the remaining two this is not so obvious. Generally it seems to depend on the lengths; given *a, b, c, d* then there are a certain number of arrangements of the sides, and in each case a different pair – but only one pair – can be made parallel. So far it has been difficult even to form a hypothesis about which pair.

There also seemed to be some theorems relating the number of right angles to the number of pairs of parallels, and we looked at which could be re-entrant. We suddenly realised we had at least twenty-three distinct quadrilaterals!

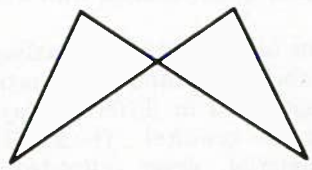
Other possibilities were raised, but not followed up at the time. A 'dual' idea was to look at angles instead of sides, but there did not seem to be suitable apparatus corresponding to geostrips, and what was the 'dual' of a right-angle? It is possible, however, to start with diagonals and classify from there. Given two diagonals, which may be equal or not (geostrips *are* useful for this exploration), what choices can be made about the relations between them, and what are the consequences?

I stress again that the *children* should make the choices, and not merely carry out a routine classification based on the teacher's choices. Decision-making is an essential part of the classification process, and it also involves making decisions about the decisions – that is, altering where necessary in the face of inconvenience, inconsistency or perhaps mere banality.

There are one or two other ways of looking at quadrilaterals without traditional restrictions. Crossed quadrilaterals are often constructed on geoboards.



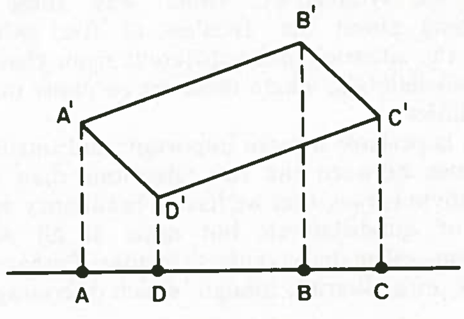
or (if they overlap the right way) with geostrips. One interesting theorem, for instance (as Cyril Hope showed us twenty years ago) is that if opposite sides are equal then they are *not* always parallel!



With geostrips it is possible to investigate which properties are preserved under crossing, and which new properties hold.

Given appropriate materials one can look at quadrilaterals in three dimensions. 'Sticks' joined at the ends with impact adhesive are suitable. What, for instance, happens to the angle sum, and what consequences does this have for a classification?

And if we can move into three dimensions we can also move into one. *One-dimensional quadrilaterals?!* Well, an attempt to investigate these with a class of 14-year-olds in front of the 200 people attending the 1965 ATM Easter Conference did not produce much, but back in the safety of their own classroom later they began working on some more precise definitions. One snag, they found, was that in one dimension *all* sides were parallel, so by the usual definition all quadrilaterals were parallelograms! Somehow they felt this ought not to be so. Parallelograms ought to come from, say, projections of two-dimensional parallelograms,



and therefore a definition in terms of lengths of opposite sides was more suitable. Here is mathematical sophistication indeed, creating new definitions in order to reconcile one's feelings about a new situation!

They discussed trapezia in a similar way, and also the awkwardness of trying to fit rectangles, squares or kites into the classification without having some coincident points.

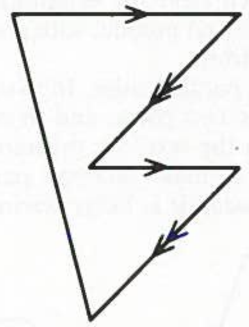
What the pupils obtained from this exploration was not any essential mathematics that was on the O-level syllabus, but a feeling of the *power* one has in mathematics to alter the variables, to examine the consequences, to recreate definitions, to make choices, and to play around with classifications. It was purely incidental, but also useful, that they went back afterwards to conventional two-dimensional quadrilaterals with a heightened awareness.

There seems to be a lot of fascinating material there, just about quadrilaterals, produced by thinking about them in different ways, freeing oneself as a teacher from Euclidean traditions – material never intended or even permitted by most syllabuses or textbooks. It requires other things too: a particular attitude towards teaching, in which geometry is something done by the pupils rather than learnt from the teacher or textbook; a belief that the processes of mathematics are at least as important as the products; and perhaps a reconstruction of one's examination arrangements to permit this attitude to be put into practice freely and the pupils' full range of mathematical abilities to be tested – for example some form of coursework. But this can be discussed later in this series.

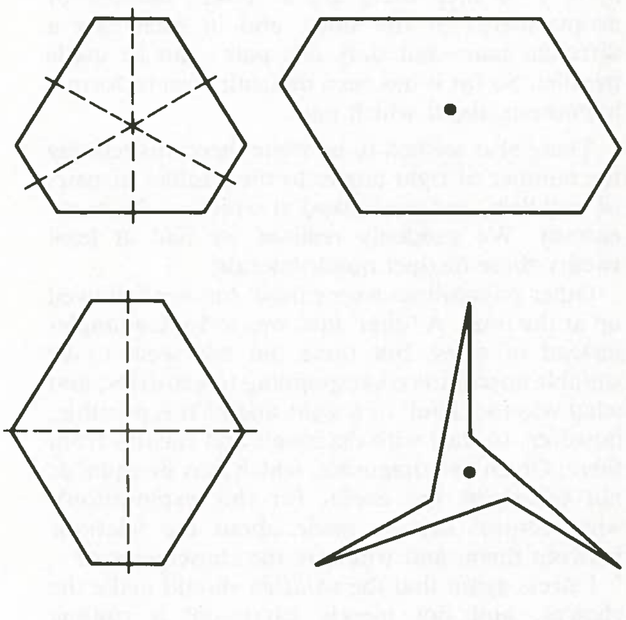
If those are some of the things we can do with quadrilaterals, what possibilities are there for releasing ourselves from the conventional syllabus and allowing ourselves to look at polygons in general?

How, for instance, can you classify pentagons? A question on a C.S.E. paper asking them to be classified according to symmetry produced from many candidates the 'obvious' three, but unfortunately with no *written* evidence of consideration of why there were only three. Like the quadrilaterals, one can consider this synthetically by drawing axes of symmetry, etc., or algebraically by looking at subgroups of the group of symmetries; either way there is something about the 'fiveness of five' which makes the situation quite different from that of the quadrilaterals, where there are so many more possibilities.

This is perhaps a more important and intrinsic difference between the two situations than the more obvious one, that we have a familiarity with types of quadrilaterals but none at all with pentagons other than regular-irregular. Perhaps it was the unfamiliarity, though, which discouraged any of the primary teachers on an M.A.Diploma course from following up a lively session on this topic and actually choosing it for their mathematical investigation. The details have been forgotten, but some ideas from a different group of teachers have been reported [4] involving symmetry, convexity, numbers of right angles and numbers of crossings. One could also look at parallel sides, completely unfettered by parallelograms and trapezia.



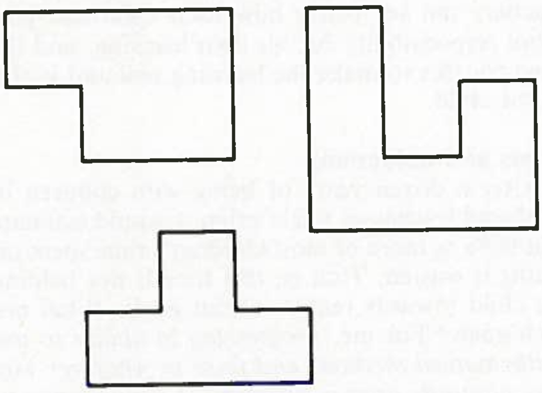
Classifying hexagons has quite a different feeling to it, not so much because there are more sides but because 6 is not a prime. This means that a classification according to symmetry produces many more categories of which the following are a few.



But, as before, there are many other possible choices.

One can look at polygons in general, in which case the number of sides becomes a variable, so that something else has to be fixed. For instance, can you define 'isosceles' for a polygon and look at the consequences of changing the number of sides?

One interesting choice has been to restrict to sides meeting at right angles. I say *meeting at right angles* because it is necessary to discuss the difference between 'internal' and 'external' right angles,



and to say 'all the angles are right angles' is not strictly true. A brief account of some work with 12-year-olds has been published elsewhere [5] indicating their ability to make classifications and produce some theorems.

There are other things to be classified in geometry, apart from polygons, and some suggestions about classifying arrangements of lines or line segments were discussed at some length in *Five Sticks* [4]. This and other ideas will be taken up in another article in this series.

The next article, however, will begin with a discussion of the problem presented in the *Editorial* of *MT* No. 91: *Given the number of sides of a polygon, what is the maximum number of right angles it can have?* Notice is given now, so that if you wish you can work at it beforehand, or better still try it out on a class.

References

[1] Philip Lewis: *Looking at Quadrilaterals* (*MT* No. 92)
[2] Geo Strips: *Taskmaster £6.95, Invicta £6.27*
[3] D. S. Fielker: *Geometry for Enjoyment* – Mathematics for the Majority Series (Rupert Hart Davis, 1973)
[4] D. S. Fielker: *Five Sticks* (*MT* No. 72)
[5] D. S. Fielker: *Strategies for Teaching Geometry to Younger Children* (*Educational Studies in Mathematics*, 10, Feb. 1979)

What is Behind the Failure?

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Pressures on teachers

I recently asked a group of primary advisers (in Australia) if, after showing a teacher how she or he could help children progress faster and more realistically, they often got the response, "Yes, but I haven't got the time, I must cover the material." There was immediate and general agreement. I then asked what unexpressed messages there might have been behind the verbal message. This they realised was a question to which they had not given much thought.

A teacher of handicapped children told me of ways she could use to get them to a real understanding. She was however teaching them directly towards tests they were supposed to pass, without quite knowing why. I asked her to picture herself, right now, as being in the classroom, teaching as she would like to. "You are teaching in this way now. What feelings are coming up?" "The door might open and the principal might walk in," was an immediate response.

The first story shows there are blocks to better teaching of which the teacher may be unaware. (She could cover the material better, catching up soon enough through better methods.) The second story shows that teachers can find and face pressures if they are helped to do so.

There are other pressures: demands of parents and public, authority of textbooks, etc. Being asked to teach in a more effective way must also be seen as a pressure. I shall look at this as a separate topic.

Immediate solutions versus gentle growth

We have all seen people expressing some problem to a group, and members of the group making suggestions, each one being followed by the response, "Yes, but . . .".

Let us put ourselves in the position of a teacher who has just seen a lovely way to let her children