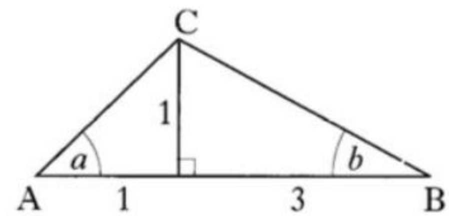
Diary: 2 October

Today we spent an hour going through some homework questions. Here are five observations about things that they did:

#1 Using inverse trig functions for questions involving unknown angles in right-angled triangles.

Both students did this last week, and again for this question:

In triangle ABC, show that the exact value of $\sin(a+b)$ is $\frac{2}{\sqrt{5}}$.



This time, however, they were aware this was not what was required (it is a question from a non-calculator exam). One student said that he found the hypotenuse of the right triangle as $\sqrt{10}$, but did not think it useful, or make the step to using it to find trig ratios. Me saying that the hypotenuse would be useful allowed both students to complete the whole question.

#2 Recognising that an idea from one context may be useful in a different one

One student was able to answer this question:

If the exact value of $cos(x)$ is $\frac{12}{13}$, find the exact value of $cos(2x)$ and $sin(2x)$.

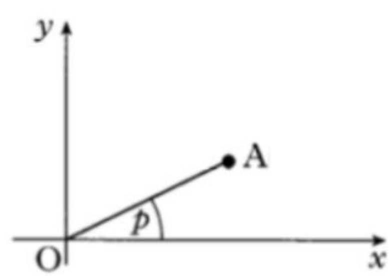
He drew a right-angled triangle, and assuming that 12 was the adjacent and 13 was the hypotenuse, had gone on the answer the question fully. I asked him how he thought of doing this, and he said: *"In the previous question [below] I realised you could create a right-angled triangle, and use it to find $\cos(2x)$ and $\sin(2x)$. I thought: what's stopping you doing the same thing here?"*

Pick a coordinate for A. The line OA is inclined at an angle p radians to the x-axis.

(*a*) Find the exact values of:

(i) $\sin(2p)$;

(ii) $\cos(2p)$.



#3 Making associations, and acting, in novel situations

Faced with this question, one of the students expanded the brackets, as he *"could not see what else to do".*

> Without using a calculator, find the exact value of
>
> $$(\sin 22.5° + \cos 22.5°)^2$$
>
> You must show each stage of your working.

This is an excellent example of doing something/anything in the absence of any other action being possible. He arrived at the following expression: $cos^2(22.5) + sin^2(22.5) + 2.cos(22.5).sin(22.5)$, but could not see where to go from there. I confirmed that this was a useful approach, and asked the students possible next steps. One student noticed the fact that 22.5 is half of 45, and the other initially suggested that we might use $cos(2x) \equiv cos^2(x) - sin^2(x)$ to transform the squared terms in some way, two partially useful suggestions.

#4 Considering / relying on one possibility for action, resulting in an inability to consider other possibilities

For the following question, a student had transformed $cos(2x)$ into $cos^2(x) - sin^2(x)$, but could not go any further:

> Solve the equation $cos(2x) - cos(x) = 0$ for $0 \le x \le 180$.

I have noticed that, *so far* during our work on double-angle formulae, he always transforms $cos(2x)$ in this way (i.e. not considering the alternative forms $2cos^2(x) - 1$ or $1 - 2sin^2(x)$).

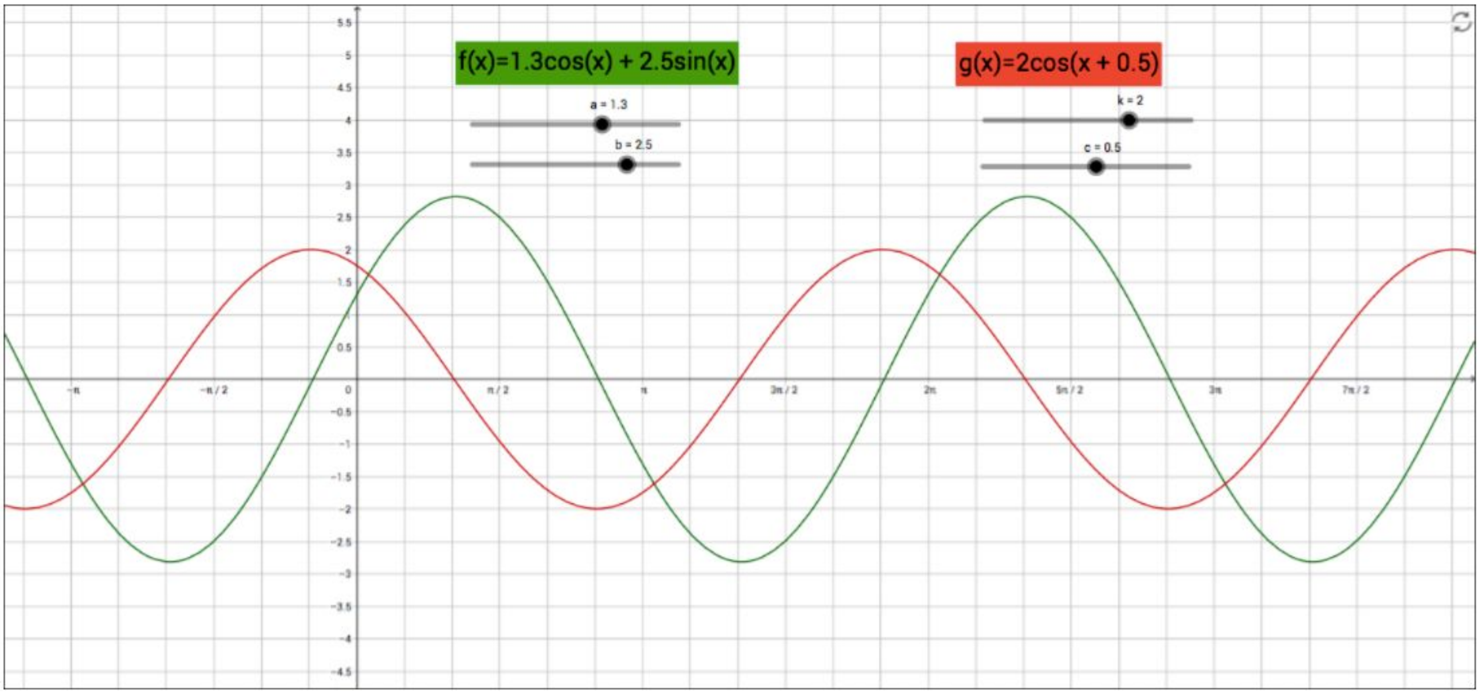
I have written about this *general* issue before, of over-reliance on one approach. It would seem that one of the jobs of teaching is to help students notice unhelpful 'habitual reactions', and find ways to engender a *freedom of thinking* that considers alternative approaches, perhaps through a 'conjecturing atmosphere'. This also requires developing a repertoire of approaches that may come to mind, in addition to noticing and reinforcing useful habits.

#5 Creating rules in the absence of knowing how to transform a (trigonometric) expression

Following the transformation described above, of $cos(2x) - cos(x) = 0$ into $cos^2(x) - sin^2(x) - cos(x) = 0$, the student then wrote: $cos(x) - sin^2(x) = 0$, because "$cos^2(x) - cos(x)$ *is the same as* $cos(x)$". During this initial work on various trig identities, both students have transformed trig functions in (similar) non-standard ways.

This might be considered a 'lack of understanding' of the 'meaning' of the functions, or it might suggest an unfamiliarity with working with trig functions in new ways (for example as algebraic objects). I do not think it is possible to 'fix' each 'incorrect' algebraic manipulation, nor is it possible to train students how to manipulate *all* trigonometric/algebraic expressions 'correctly'. I suspect what is required is to encourage students to find ways of checking whether a certain transformation 'makes sense', if they are unsure about it.

After working on the homework, I asked students to sketch the graph of $y = sin(x) + cos(x)$, before working on this [this geogebra applet](). Here is a screenshot:



Say what you see.

What questions do you have?

Can you answer your own questions?

I asked them to *say what they see,* here are some of their responses:

- *Changing c moves the [red] graph left or right, and changing k changes the height.*
- *Changing a and b or both changes the [green] graph's height and length.*
- *a moves the start point of the cos graph on the y-axis.*
- *b moves the turning point of the sine graph.*

I then asked them to write down any questions they had, which included:

- *Does* $f(x) = 1.cos(x) + 1.sin(x)$ *start at 1 and have 1 cycle in* 360° *?*
- *What does* $g(x)$ *have to do with* $f(x)$ *?*
- *How much do* a *and* b *affect the graph?*
- *How many cycles in* 360° *does* $f(x)$ *have?*
- *How could you work out* a *and* b *if you didn't know the values [presumably, given the graph]?*

We then worked on these questions, identifying that $f(x)$ always had period 2π (contrary to one conjecture that the length was changing), and confirming that it was easy to predict the effect of changing the parameters of $g(x)$, but not those of $f(x)$. Questions 2 and 5 were fundamental to the work I had planned to do, as transforming expressions of the form $acos(x) + bsin(x)$ into the form $kcos(x - \alpha)$.

Interestingly, while playing with the applet, they had experimented with making $f(x)$ and $g(x)$ the same height, but had not experimented with making them overlap, as a previous learner I used this task with had done.

This awareness, that the two graphs can always be made to overlap, is a fundamental aspect of the task. Eventually, I set some (random) values for a and b, and asked them if they could change k and c to make the two graphs overlap.

I then asked if they thought this would always be possible, for any a and b. They thought it would be, giving this lovely rationale: *"You can change k to make $g(x)$ the right height, and then use c to shift it into the right place."*

This provided a nice visual rationale/representation for introducing the algebraic procedure for transforming expressions of the form $acos(x) + bsin(x)$ into the form $k.cos(x - \alpha)$: we first find k, and then α. I then demonstrated the procedure for such a transformation, and they practised a few of examples with a view to making the procedure automatic.