COUNTING COUNTS

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I want to offer a story about arithmetic. This won't be history, but my story. It doesn't have to be your story, but I think it is important you have one, and that we should exchange our stories. The stories that we tell each other greatly influence what we do.

It is my privilege to start. But before doing so I give myself a warning shot across the bows from Philip Ballard, who wrote an influential book on the teaching of arithmetic in the year I was born. His preface starts as follows (with the then customary masculine nouns).

> Rarely is a teacher satisfied with the arithmetic of his class. He is not allowed to be. Somebody always has something to say about it — generally something unpleasant. If it is not a colleague, it is probably an inspector; if it is neither of these, it is a parent or an employer. And even when the criticism is not aimed at a particular person, but lies couched in a general report on a public examination, or in a survey of a prescribed area, or in an utterance from the platform or the press, its generality makes it nonetheless a disturbing force... What the teacher has to do is grow a skin thin enough to let him know when he is hit, but thick enough to protect him from serious wounds.

Current criticism of the teaching of arithmetic pre-supposes some widely accepted points of view. For example, one of the winners of a recent ***Guardian*** competition for a curriculum for the next millennium was a schoolgirl who suggested pupils should be taught mathematics in terms of mortgages, banking, shares and so on. This was consistent with her other suggestion that they should learn baby-care and other parenting skills. It is, of course, not a new notion that schools should be a preparation for life. The question is what sort of life? And it is always worth recalling the comment by E M Forster in one of his novels that the issue is not always that of being taken unprepared, but rather that of being prepared but never taken.

The traditional tale told about arithmetic is certainly very much related to its applications in the adult world. Indeed the telling of tales reminds us of the bank teller who counts our cash, and also of the tally with which people used to count various things. Tales are also accounts. A common historical account of the development of numbers refers to the raising of taxes in ancient Egypt. And the 'accounting' I am invoking reminds us that the burst of arithmetical activity in renaissance Europe began with the counting of money and other assets.

Before the new-fangled Arabic numerals were widely adopted, numerical calculations used counters placed between lines marked on a board or table. This was a form of abacus, sometimes called a checker-board, whence our words exchequer, cheque. (Compare the German *rechenbanck*, whence our words bank and bankrupt – literally a broken board, and note we still buy things 'over the counter' in shops today). People were very skilled in the use of these boards and it took some time before they adapted to the new numerals. Some of our current difficulties in the teaching of arithmetic echo the problems that arose in Europe as new arithmetical procedures were being developed.

Here is a woodcut from one of the earliest printed books – a popular encyclopædia by Gregor Reisch, first published in 1503. The presiding woman is *Typus Arithmeticae*, the spirit of Arithmetic which was one of the seven liberal arts of the classical curriculum. Pythagoras is shown on the right manipulating counters on a counting board. A skilled practitioner of this form of abacus was known as an *abacist* and would have been in great demand in the early days of commerce. The figure on the left who is using the new numerals is Boethius, a sixth century Roman philosopher who had been the main source for the mediaeval accounts of Greek mathematics. Methods of calculating with these numerals were called *algorisms* (or algorithms – the word derives from the name of the ninth-century mathematician Al-Khwarismi) and a person who calculated directly with the new numerals was called an *algorist.*

Abacist and algorist

The picture is often interpreted as showing a contest between the old and new methods of calculating, with the spirit of Arithmetic favouring the algorist over the now outdated abacist. But there are some tantalising ambiguities to be found in the picture that suggest something more complicated than just a simple binary opposition between successful innovation and defeated tradition. I feel that these ambiguities are still at the heart of many current problems in the teaching of arithmetic.

To put it briefly, Pythagoras' counters may be said to be 'concrete' representations of amounts of coins, goods, or whatever. This is historically appropriate in that it was in the Pythagorean tradition to represent numbers by pebbles. This is not unlike the way children are encouraged to work in primary schools. Well designed and carefully planned activities offer children metaphors for numbers in the form of substitutes – such as counters, coins, rods, or mental images of these – which are understood to behave like numbers in some respects.

This is also not unlike what is naturally encountered at home or in the streets and shops where numbers always refer to amounts of things. Except for the fact that the classroom activities are inevitably artificial and not only do not really matter, but are explicitly understood as not really mattering. Thus, Valerie Walkerdine quotes an incident in which an infant teacher joins a group engaged in a prescribed 'shopping' activity, and asks a boy to buy a dolly for her. He says he has only one penny left. Another child points to a pile of coins, but the boy says that he had already spent those. The teacher then comments: "Oh, I see what you are doing. No it's alright,. . . each time, you get another ten pence to go shopping with." Counting pennies may have different meanings – for some a grim reality, for others a casual game of let's pretend. Using money as a metaphor for number in classrooms may be sometimes problematic.

The algorist may also be said to be manipulating (handling) concrete representations of number in the form of the new numerals. It is strange that it is Boethius who is supposedly the innovator, for he had been specifically Pythagorean in his approach to arithmetic. His unlikely role as an algorist is ignored in the rest of Reisch's book. This presents arithmetic in two parts. The contrast is not in fact between algorist and abacist, but rather between the so-called 'speculative' arithmetic which was a direct summary of Boethius (and what we would now think of as number theory), and 'practical' arithmetic which included the new methods of the algorists as well as a traditional treatment of computation using counters. Speculative arithmetic was of no interest to merchants and bankers. But practical arithmetic, whether carried out with counters or pen, was of no interest to Greek or Roman philosophers. It remains surprising that Pythagoras and Boethius were cast as calculating practitioners, and supposedly opposing ones at that.

I shall be asking you to see the Reisch picture in terms of complementary, rather than opposite approaches to arithmetic. But first you might like to see an amusing contemporary version.

The abacist is still named as Pythagoras, who appears as a somewhat baffled senior manager. The algorist is an anonymous and self-satisfied yuppy. Significantly enough, Arithmetic is now male – a bowler-hatted, pinstriped City gent with a rolled umbrella who is here giving neither of the others a glance, but seems to be using some sort of sextant. The moral of the picture is as ambiguous as the original version: we might like to read it as part of the current debate about calculators. And isn't that a laptop on the floor?

Metaphor and metonymy

The new numerals were associated from the start with the market place. But they were a step away from the material reality of counters and coins, and the actual material goods and physical labour that lay behind those. This distancing – a shift of the signifier from the signified – is a characteristic feature of the subsequent development of mathematics. In parallel, some might say, with the development of capitalism.

The new numerals began to take on a life of their own. They were – and often still are – held to represent numbers that are supposed to have some reality independent of their representation. But this distinction between numeral and number is unfortunate, and can often be the source of much confusion. This issue was amusingly raised by Magritte in his painting of a pipe with the caption 'Ceci n'est pas une pipe'.

Access to the 'numbers' supposedly represented by the new numerals can only be through symbolic operations on paper. Unless, that is, substitutes are offered like the counters, blocks or rods provided in

classrooms. But in this case, of course, the access is to the number world of the abacist, and the problems of the algorist may still have to be tackled.

I associate the number world of the abacist with the figure of speech known as metaphor. We regularly invoke metaphor when introducing mathematical concepts: numbers are said to be like lengths, negative numbers like debits, functions like curves. But mathematics itself seems to need to get away from metaphor as soon as it can. For example, odd and even were originally defined in terms of actual physical separation of a pile of stones. But the actual division soon becomes a virtual operation in the mind. An even number may then be defined as a number that can be divided into two 'exactly'. A further shift in the concept occurs when even numbers are simply defined as multiples of 2: attention is now drawn to which numbers are even, rather than what evenness is. This also shifts from the inverse operation – actual or virtual – of division to the direct one of multiplication. But it does not invoke the multiplying so much as the results of doing so – a lingering trace in the memory of a chanted chain of signifiers: two, four, six, eight . . . (whom do we appreciate?).

This is to emphasise the figure of speech called metonymy (literally, a change of name). Metaphor shifts to another notion in terms of some sort of likeness: equations are like balances, some say. Metonymy shifts in terms of some sort of contiguity: this may be an actual neighbourhood as when referring to a bottle of Bordeaux, or a part-whole relationship as when referring to a book by its author ('pass me the Shakespeare'), or a contextual link as when a musical note follows another, or for that matter when mathematical symbols are brought together in equations. In effect, the equivalence notion in mathematics is metonymic when it is interpreted as 'another name for' (thus, another name for two-plus-two is four). Algorithms involve a sequence of metonymic substitutions. So I associate the number-world of the algorist with metonymy.

Metaphor and metonymy are not necessarily distinct polarities, but more like aspects that can be stressed or ignored as desired. One of our problems in teaching arithmetic is the move from a stress on metaphor to a stress on metonymy. We offer children counters and rods and so on, in order to mimic processes which we eventually want them to transfer to written or spoken numerals. There are various eloquent accounts of the sometimes unexpected difficulties that this may cause. Here is an encounter between Anna, the amazing heroine of the book, ***Mr God this is Anna***, and her teacher who has set her addition problems in terms of sweets.

> *You have seven sweeties in one hand and nine sweeties in your other. How many sweeties have you got altogether?*
> Anna: None . . . I ain't got none in this hand and I ain't got none in this hand, so I ain't got none, and it's wrong to say I have if I ain't.
> *Her teacher tries again: I mean pretend dear, pretend that you have.*
> Anna: Fourteen.
> *Oh no, dear . . . You've got sixteen. You see, seven and nine make sixteen.*
> I know that, but you said pretend, so I pretended to eat one and I pretended to give one away, so I've got fourteen.

The same point was once made by Mary Boole who emphasised that a quarter of three apples is not quite the same as three quarters of one apple. She pointed out that the metaphor for division as sharing can lead into some difficulties. How do we divide a cake weighing five ounces between two children? Two and a half ounces? But do we give one child all

M: Twenty-eight. What's thirty and two?
Thirty-two.
M: Great. Should we get up now?
Yeah.

Temperence, by Pieter Bruegel the Elder, 1560

Time and space

Now, as a slight digression before I come to the main theme of my story about arithmetic, I would like to show another sixteenth century woodcut, this time one made by the painter Pieter Bruegel. It is taken from a series depicting the seven virtues, in this case Temperance, and serves to remind us that not all the mathematical developments of the time arose in the market place.

Starting at the top right we see astronomers measuring angular distances in the sky and on the globe. Then, working clockwise, there are masons with various instruments. Theologians and philosophers dispute in front of the artillery that settles other arguments. A class is learning to read - reminding us that the alphabet and spelling were just beginning to be standardised at the time. Over to the bottom left, we see an abacist at his counting-table and an algorist working on paper, as well as one who seems to be jotting on the back of a lute. Behind them is a painter, reminding us that it was the renaissance artists who developed the rules of perspective that were much later to be mathematised as projective geometry. Moving up on the left are the musicians who were developing polyphonic music and the new notation required to record it. Above them is a stage – Shakespeare and Lope de Vegas were born at about the same time as this woodcut was made.

In the centre is Temperance, with various symbols of her supposed virtue: sagacity represented by the spectacles in her left hand, self-restraint by the reins in her right hand which curb the bit in her mouth; control by spurs on her heels and by the snake belting her waist. She stands on part of a weather vane – the important achievement of mediaeval technology. Most significantly, her headpiece – right in the centre of the picture – is a clock, the powerful new technology that measured time accurately.

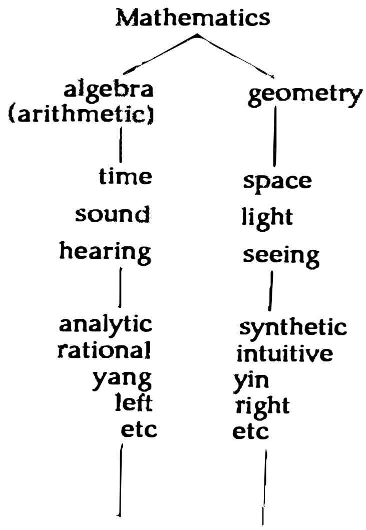
The picture shows many people engaged in one way or another in visualising the stuff of reality in terms of uniform units such as leagues, degrees, musical notes, numerals and letters, hours and minutes. Here is a medley of images from the Renaissance dream: that yearning for order which found expression in new ways of visualising space and quantifying time. We are reminded that mathematics has its roots in many other practices than commerce and banking.

In particular, I take the geometry and algebra of any particular culture – or period of history – to be its current mathematisation of space and time.

> time and space are real beings
> time is a man and space is a woman
> (Blake)

> space is a swarming in the eyes
> and time a singing in the ears
> (Nabokov)

Perhaps Blake would not mind me seeing his gender distinction as referring to the male and female in each of us. The second quotation leads me to suppose that time may be associated with hearing in the way space can be associated with sight. I can then offer a story about mathematics in the form of a diagram.



I have discussed the right-hand side of this diagram at a previous conference in Cumbria (see MT129). The left-hand side condenses my story about arithmetic. It points to an initial emphasis on repetition and rhythm. This leads, for me, to numbering in a general algebraic sense: hence the notion that algebra precedes arithmetic.

of the top and the other all of the bottom? Don't we have to share the icing and the cherries if we want to avoid arguments?

I am sure you will have come across the issue in your own classrooms. Exchanging ten unit rods for a single rod is not the same action – or accompanying thought – as is involved in written work when 'carrying' a unit numeral to the next column. The algorist is not reproducing the actions and thoughts of the abacist. Thinking of negative numbers as temperatures isn't much use when adding them; thinking of them as debits isn't much use when multiplying them. Thinking of functions as graphs gets you so far, but eventually has to be replaced by a sort of culinary metaphor in the form of a rule or recipe; then that too tends to be replaced by an actual table of values – a metonymic chain as in the 'extensional' definition of evenness as the actual set of even numbers.

Early learnings

From one point of view, the move towards metonymy is sometimes seen as the terrifying abstract price that mathematics demands. The apparent decontextualisation of number work conjures up traumatic memories of formal teaching, rote-learning, mindless mind-training, and all the other bogeys of yester-year. An emphasis on the actual social contexts in which numbers are used - the very point which the winner of the *Guardian* competition was urging – is encouraged by a Piagetian psychology that conceives of abstraction as a formal thought process that takes many years of development to achieve. And abstraction can sometimes be seen as a terrible denial of reality and a repression of feeling, so that mathematics becomes yet another tool in the oppressive patriarchal order. And so on. Yet, mental and emotional reality may be otherwise, and there is accumulating evidence that babies are involved in abstract processes from the very start.

Caleb Gattegno emphasised the central importance of the essentially abstract notion of equivalence in the early development of perception and of language. Any parent will be able to provide many examples. I recall walking in town with a friend and my son. The adults' conversation was suddenly interrupted by my son who had noticed the new department store in the distance. "Look, Daddy, there's Bobby's". "Yes, yes, George", the preoccupied father replied and continued talking with his friend. A little later, after a few turns and crossings, my son suddenly got very excited. "Look, Daddy, another Bobby's". At last I attended, and was struck by recaptured perceptual innocence. All the views of buildings – from different positions, in different lights, in different moods – are indeed different. Yet, for various reasons, we do need to be able to see them also as the same: there's Bobby's. Different perceptions are gathered together into an abstract single entity – an equivalence class. Learning the use of nouns is an early mastery of abstraction, an exercise in algebra. And, as Gattegno emphasised again and again, we would do well to work with such powers as those that children already own. It could be economic for us, as well as the children.

Most children experience playful ways with words, most obviously with the actual number-names which they pick up from various rhymes and counts before they ever consider counting actual objects. Again most parents will have plenty of examples. I have recently been listening to my three-year-old twin granddaughters. They name numbers with reckless abandon, sometimes partly in order. The one who seems already to have mastered language is interested in relationships – grannie is mummy's mummy. She will usually say the number-names in correct ascending order and is tickled by the game when I suggest she says them backwards. The other twin is less secure with language: she is interested in things and how they work – what happens when I do this? (often something disastrous from an adult point of view!) But she is also fascinated by reciting number-names – she rarely gets these in order, partly because she always inserts another two now and again. This makes sense for someone whose world is very much twice.

I owe some further examples of metonymic play to a Canadian teacher, Eileen Phillips, who has carefully recorded various conversations with her daughter in a master's thesis. Here she is playing a game with her daughter, aged 4:3, who starts by asking her mother to add two numbers.

What's eight and eight?
Mother: Sixteen. What's three and two?
Five. What's ten and ten?
M: Twenty. What is one and two and three?
No. Go 'what's m and m'.
M: Okay. What's three and four?

"Go what's m and m". What an impressively economic – algebraic – way of getting mother to stick to the game and adding only two numbers. Here is another example, with the daughter, now aged 5:5, joining her mother in bed in the morning.

What's twenty plus three?
M: Twenty-three. Why?
I want to play that game.
M: Okay. What's twenty plus six?
Twenty-six.
M: Wow! How did you know that?
You just did it, so I copied you.
M: So, twenty and three is twenty-three, so twenty and six is twenty-six.
Yes. What's twenty and eight?

TIME	SOUND
rhythm	ordering names
repetition	naming orders
numbering	counting orders
. . . . .	ordering counts

(At the conference I displayed the gist of the last few paragraphs on the overhead projector – to the accompaniment of the second movement from Haydn's symphony no. 100, known as the Clock because of the steady tick-tock rhythm of this movement.)

I suppose we could say we are born with rhythm, that steady beat of the heart that will be with us until the day we die. This is our internal clock that marks the passage of time. But very soon we find we can objectify this internal awareness. Babies bang cups on tables, they can open and shut their eyes at differing rates, and lying on their backs they kick their legs repeatedly in the air. And they can make noises and clap hands – and do so repeatedly. As Gattegno emphasised, when you can say 'ah', you can also say 'ah-ah'. But you can also vary the rate of repetition to create different rhythms. Part of the delight in learning the number names is that you can say them musically – and conversely this is the best way to learn them. One, two, buckle my shoe; three, four, knock at the door . . . One, two, three, four, five; once I caught a fish alive . . . And all this happens long before you count the number of things – a sophisticated process which may take some time to be mastered.

I am here drawing your attention to two ways of counting. There is counting intransitively – 'just' counting, or merely mouthing the words as some may be tempted to say scornfully. And there is counting, transitively – counting objects in order to name how many there are. The distinction may sometimes seem unimportant, but it does, I suggest, crucially affect our practice at all levels of schooling.

Ordinal and cardinal

Is counting in either of its aspects learned by imitation of those around us? Some linguists have called attention to the fact that in learning to speak children cannot be said to be learning by imitation. They can, and do, make sentences that they may have never heard anyone else say.

– It's time to go up to bed, George. Daddy, when is it time to go *down* to bed?

– A girl says, I *eated* up all my sweets yesterday. Her sister replies, Oh! I *ated* mine too.

So it is in numeration. We are given some number-names through nursery rhymes, counting-out games, skipping songs, and so on. We hear and copy some special forms like eleven, thirty, hundred, but when a child says 'one-hundred and seventy-two' it may well be that this has never been heard before. Moreover, it is a relatively easy task to say what comes after 172, or to go on counting from there. (If you know that 3 follows 2, you know that 173 follows 172 – and that's for free!) Being able to say the words one to nine in order, and being able to say -ty, hundred, thousand, or million at the right places means that you can say literally millions of numbers which you will not have heard before. How are you able to do this? What is the mind working on when it generates numerals in this way?

It is surprisingly easy to count in sequence up to 172 and beyond, without any objects to match with the spoken numerals other than the required acts of attention. The recitation of number-names in order illustrates the ordinal aspect of number. It reminds us that in counting, say fingers, you say nine when you get to the ninth finger. You have been noting the fingers that come first, second, third and so on. "Show me three with your fingers", I ask my granddaughters. One of them puts up one finger – the third that she had counted. The other puts up three fingers, and this reminds us that in counting sets we attach the final (ordinal) number of the count to the set as a whole. This then gives us the number of objects in the set – the so-called cardinal number.

The distinction between ordinal and cardinal numbers doesn't really matter in usual day-to-day calculations. But it does concern mathematicians who have disagreed about which process is more fundamental in the logical development of arithmetic from first principles. It also concerns psychologists who disagree about which process comes first in the psychological development of children's understanding of arithmetic. Moreover, teachers also have to take some view, explicitly or implicitly, because it will affect their practice.

These three different areas of concern – mathematical, psychological and pedagogical – have interacted in various ways in the past. But it does not seem to me that they need come to the same conclusions. The fact that we can switch from one sort of number to another does ultimately depend on the important property that, no matter what order you count your fingers, beads, blocks, coins or other objects, you will always end up with the same number. For the mathematician this is a theorem to be proved, for the psychologist a concept to be acquired, for the teacher an awareness to be educated. As you will know, it is not always immediately obvious to young children, and much time is (or used to be) spent in infant schools on experiences which would lead them to master the notion which Piaget called conservation of number.

Piaget himself was fairly neutral on the vexed question whether ordinal or cardinal notions come first in the development of children's understanding:

he claimed that the notions were subtly intertwined. Nevertheless, because his conservation experiments were so startling when they were first announced, many of his followers have emphasised cardinality. So, for example, many English primary schools gave up the chanting of multiplication tables, which are in effect but the chanting of certain sequences of ordinals. Psychologists are still divided about the issue as indeed are mathematicians in the case of the corresponding problem in the philosophical foundations of arithmetic.

Mathematicians did not at first have any need to make the distinction between cardinals and ordinals, leaving this to grammarians and lexicographers. The distinction was forced upon them little more than a hundred years ago by Georg Cantor's work on infinite sets. Cantor found that the conservation property didn't hold for infinite sets. It did matter what order you counted in. You could count infinite sets in different ways and get differing results. The conservation property turned out to characterise finite sets, the corresponding theorem being that a finite set – unlike an infinite one – can be associated with a unique ordinal. Cantor had to find some other way of defining and determining the cardinal number of an infinite set. He was so taken by his discovery of a way (one-one correspondence) of doing this that he took cardinality to be fundamental. In turn, Piaget – who was very much interested in and influenced by work in the logical foundations of mathematics – took one-one correspondence to be fundamental in the psychological foundations of children's arithmetic. And so children end up with those dreary exercises in drawing links between pictures of cups and saucers, birds and numerals, whatever.

Counting counts

"Them as counts, count moren than them that dont count", says the eponymous hero of Russell Hoban's novel, ***Riddley Walker***. And you will have gathered that I think it is ordinal counting that counts.

There have been mathematicians who have taken a different point of view than Cantor. For example, according to his friend Richard Dedekind,

> the whole of arithmetic [is] a necessary, or at least natural, consequence of the simplest arithmetical act, that of counting, and counting itself is nothing else than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding.

And this is the basis of the so-called Peano axioms. There have been many psychologists who have similarly emphasised ordinality.

Moreover, there are other views of the historical foundations of number. Not all historians take the view that numbers first developed in order to count property. For instance, Abraham Seidenberg has proposed a ritual origin for the number sequence. This is envisaged as consecutive calls of the names of the gods represented by participants in a ritual. As if to say, come in One . . . now it's your turn, Two . . . Three . . . and so on. Hence the march of the integers: 'rite words in rote order'.

This story might not be to everyone's taste. But, for me, it is not so much a matter of what was actually the case, rather a matter of what stirs and inspires. History is always in the present. We can always choose what to stress and what to ignore. Here and now, I am choosing to stress metonymy rather than metaphor, ordinals rather than cardinals, communal counting rather than individual calculation.

And I am also invoking the thought of a magical moment when someone realises that you can't just go on giving gods individual names. Unlike, you may recall, that character in a story by Jorge Luis Borges who devised a new system of numeration in which every number was given a special name of its own. For example, for 7,013 he would say Maximo Perez, for 7,014, The Train, and so on. At some stage you would run out of memory or patience. As Borges comments, 'to think is to forget differences, generalise, make abstraction'. So someone would have to devise that magical cyclic structure that lets you count on and on with very few new names.

Let me illustrate some of this by briefly encroaching on work that I imagine some colleagues will be offering later in more detail. Here is an idiosyncratic version of a counting array proposed by Gattegno.

J	A	J	W	U	T
JH	AH	*	*	*	*
JHH	*	*	*	*	*

This is designed to ease access to the underlying structure of an ordinal count. I use a nursery rhyme to mimic the situation in which very young children come to know the first few number names in the right order. One, two, three . . . But the structure can just as easily be seen with A, B, C . . . or Jack And Jill Went Up The . . . I know you know the next word but that is in effect my number base and it appears in the second row. Now you could count following my pointer which might tell you to say *jackhill, jackhill-jack, jackhill-and* . . . and you might then be able to carry on.

Well, that could be too easy, so you might start again missing out every other call: *jack, jill, up, jackhill*... What about starting *uphill* – and counting backwards? . . . Or counting the went-times table? This would be the multiples of *went*, so the chant would start with *went* and then call the next *went*-th number (namely, *jackhill-jack*), and so on. Multiplication tables are but syncopated counting. Working with such number arrays is easy. Yet they also present some challenges. Mastering them is an essentially algebraic business. But I suggest it pays off enormous dividends. The process seems to me to be quite fundamental. It is a group activity demanding individual mental work. One soon moves away from using the table and begins to invoke various patterns as they arise. I would want any group I teach to be able to chant communally (and I would want to emphasise the unison) through various appropriate arithmetical progressions. "Starting at 1089, let's count backwards saying every seventh number". What, I would want to know, would be the point of doing any other number work with students who couldn't do that?

Much traditionally expected number work follows quite naturally from all this. The customary cardinal emphasis leads you to read 3+2=5 as a statement about the cardinal number of a union of disjoint sets. In school you might at first be asked to count out three blocks and then two blocks, then you shove them together and count the result to give five blocks. Later, you will read it as 'counting on' two places from the name three in an ordinal chant. And there is no reason why you could not be doing this in the first place. You get a lot for a little by milking this second reading for all it is worth: in fact you get the four rules of arithmetic, as Ballard pointed out nearly seventy years ago.

> The four fundamental processes in arithmetic are merely different ways of counting. Adding is counting forwards, and subtracting counts backwards. In multiplying or dividing we count forwards or backwards by leaps of uniform length.

Linearity

Counting forwards or backwards in uniform leaps might be seen as a natural and early example of linearity. (I stress this in case it may seem that I have been mainly addressing primary work). I have always been puzzled by the fact that adolescents are required to work through example after example of linearity each of which can often seem to them a new topic.

- If this many costs so much, that many costs how much?
- If so many shekels are worth so much, how many for this much?
- This fraction is equivalent to what percentage?
- If this recipe is for so many people, what do you need for this many?
- How long do I take to travel that distance at this speed?
- Given the perimeter of that circle, what is the perimeter of this circle?
- Using sine tables, what is the length of this side of that triangle?
- Henry VIII had six wives, so how many wives had Henry IV?
- ....

Costing exercises, measure conversions, equivalent fractions, ratios and scales, rates and averages, perimeter problems, trigonometrical ratios... the list almost entirely covers the secondary arithmetic syllabus. Teachers will recognise that these topics are related. But do students realise that they all tread the same water? What is the mathematical awareness behind all these forms? When – and how – is it first acquired? How is it that after more or less ten years of practice in such isomorphic exercises, so many can still not perform them satisfactorily? Why is it – precisely – that having covered one of them, the next manifestation is not immediately accessible? Is failure more often due to lack of concept – or is it due to lack of control?

Well, it is easy enough to ask questions. But that's my present privilege! I repeat that counting forwards or backwards in uniform leaps might be seen as a natural and early example of linearity. Who would not be happy to work with a class which had mastered that, even if nothing else?

"Starting at 1089, let's count backwards saying every seventh number". What, I would want to know, would be the point of doing any other number work with students who couldn't do that?

Note

I wish to acknowledge that my remarks about the Bruegel woodcut owe much to Alfred Crosby's remarkable book, ***The measure of reality*** (CUP, 1997). I also note that part of the discussion of the Reisch woodcut is adapted from my chapter in the OU reader, ***Teaching and learning in school mathematics*** (Hodder & Stoughton, 1991).

The cartoon version of the Reisch woodcut is reproduced by courtesy of the artist, David Cousins, who drew it for an issue of ***The Economist*** in 1979.

The Magritte cartoon was specially drawn to accompany this article by Tom Hoare.

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PETER LACEY'S TEN QUESTIONS

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