Examples

Before reading on, take a few moments to answer this question:

What comes to mind when you consider the (mathematical) concept of a straight line?

Now suppose that I add to this request that the ‘domain’ is the secondary school classroom; what comes to mind now?

The image that came to my mind is a line of gradient 2-ish, crossing the y-axis just above the origin. The algebraic example that comes to mind is $y = 2x + 1$. There is also for me the sense that the line can rotate (change gradient) and shift, accompanied by variation in the equation.

Did you have similar images or examples come to mind? Perhaps you were influenced toward creating examples by the title?

Imagine now that the domain of interest is the A-level, or Higher, maths classroom...

I wonder how the space of examples you have in mind has now expanded? Do you have more images and/or examples? Or perhaps some other algebraic structures?

Now: what examples might the students that you teach have available? Is their example space rich and flexible enough to solve a wide range of more or less familiar problems, such as those they will face in an exam?

When introducing a concept, a teacher might present an example that has served them well as a *reference example* (a single example that represents some generality) for their students .

What are the advantages and disadvantages of presenting $y = 2x + 1$ *as a reference example for students encountering straight lines for the first time? What are the alternatives?*

Following the presentation of some examples in some form, the teacher might then attempt to create a situation in which students come to ‘understand the meaning’ of the constants in the equation. There is an often dizzying range of pedagogical choices available to the teacher.

Each pedagogical choice provides the opportunity for more or less choice on behalf of the student. There are situations in which more or less choice may be preferable. But without the exercise of initiative in some way, it is unlikely that the learner will gain much from the experience.

In *Mathematics as a Constructive Activity*, John Mason and Anne Watson describe the importance of providing opportunities for learners to exercise initiative by shifting the responsibility for example creation from the teacher to the learner, *‘... from making sense of examples, to creating examples to make sense.’* (p.8).

Most of the time when doing mathematics, students are not learning a concept from scratch. They already have a set of examples that they bring with them. There are various *local generalisations* that may have been useful (and indeed necessary) in previous domains, but which may make extension of the example space difficult.

An example of such a local generalisation might be 'multiplication makes bigger'. This will be true for nearly all examples of multiplying in the formative years, the domain being the natural numbers, coupled with image of multiplication as 'groups of'. Students may disregard the 'non-examples' of multiplying by 1 or 0, should they encounter them, which may make future expansion of the example space difficult.

What local generalisations about straight lines might be implicit in the example $y = 2x + 1$*?*

Students may disregard examples that don't fit a generalisation because they are considered odd or inconvenient. There is the requirement for economy when learning, and the cost of these non-examples might not be considered 'worth it'.

What examples of straight lines do not fit the general form suggested by $y = 2x + 1$*?*

Students often find it difficult to accommodate the whole range of examples of straight lines such as $x = 2$ and $x + y = 5$ into their existing conception of a straight line, based as it is on the form $y = mx + c$.

One way a teacher may enable students to make sense of such a wide range of possible examples is through providing students the opportunity to create difficult, interesting or unusual examples of problems, which they can then work on together.

There may also be a case for showing multiple representations of the mathematical object under consideration. Why not consider the form $ax + by + c = 0$ when introducing straight lines?

What might be the advantages/disadvantages of introducing students to the form $ax + by + c = 0$ *as an alternative to* $y = mx + c$*?*

How can the teacher know something about the example space learners bring to a situation? It is not possible to know how anyone's example space is constructed, but one can observe what examples come to learner's minds in certain situations.

One approach is to use a diagnostic test. A less threatening approach is described by Malcolm Swan in *Collaborative Learning in Mathematics* (p.88):

'An initial activity is designed with the purpose of making students aware of their own intuitive interpretations and methods. At the beginning of a lesson, for example, students are asked to attempt a task individually, with no help from the teacher. No attempt is made, at this stage, to 'teach' anything new or even to make students aware that errors have been made. The purpose here is to expose pre-existing ways of thinking.'

Another approach might be to use tasks - such as the matching tasks designed by Swan - in which the first attempt is clearly visible, but in which conjectures are easily modified by simply moving a card from one place to another.

A simple approach might be to ask students to list as many different 'types' of mathematical object (i.e. a straight line) as possible, and then revisiting this at a later time, hopefully with the result of subsequent learning activity being the expansion and (re-)organisation of their example spaces.

Why the emphasis on creating rich, flexible example spaces? At the least, teachers must create situations in which students can develop an awareness of variation and structure in the mathematical objects, and the connections between them, on which they are to be examined. This rich space of connected examples must come to mind when faced with questions that are invariably 'non-standard' in some way.

As an example, here is the first question from the 2016 Scottish Higher exam:

> Find the equation of the line passing through the point $(-2, 3)$ which is parallel to the line with equation $y+4x=7$.

I wonder what students must have felt when faced with this question? For some it may have felt familiar, for others, less so.

What must come to mind in order to be able to answer this question, and all of the other questions that could have been asked?

Clearly there are a range of examples - and connections between examples - that must be internalised if someone is to be able to answer such questions successfully, alongside a confidence from (creating and) working on unusual problems.

Integrating unfamiliar examples and structures into those held previously requires creative teaching approaches, particularly where there may be cognitive conflict.

The learner must themselves become aware of how the new examples fit in with, and expand upon, what they already know. There must be time to reflect, to re-organise their example spaces.

Central to this is the requirement for teachers to provide opportunities in which learners are encouraged to exercise initiative; a *'shift from making sense of examples to creating examples to make sense.'*